

# THE 2/3-CONVERGENCE RATE FOR THE POISSON BRACKET

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**Abstract.** In this paper we introduce a new method for approaching the  $C^0$ -rigidity results for the Poisson bracket. Using this method, we provide a different proof for the lower semi-continuity under  $C^0$  perturbations, for the uniform norm of the Poisson bracket. We find the precise rate for the modulus of the semi-continuity. This extends the previous results of Cardin–Viterbo, Zapolsky, Entov and Polterovich. Using our method, we prove a  $C^0$ -rigidity result in the spirit of the work of Humilière. We also discuss a general question of the  $C^0$ -rigidity for multilinear differential operators.

## 1 Introduction and Main Results

### 1.1 Lower semi-continuity of the uniform norm of the Poisson bracket.

The present note deals with the  $C^0$ -rigidity phenomenon of the Poisson bracket. More precisely, for a symplectic manifold  $(M, \omega)$ , we have a notion of a Poisson bracket  $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ . For given  $f, g \in C^\infty(M)$  and a local coordinate chart, this bilinear form involves partial derivatives of the functions  $f, g$ . Therefore, we have no control of the change of the values of  $\{f, g\}$  when we perturb the functions  $f, g$  in the uniform norm. However, it turns out that when we restrict ourselves to compactly supported functions on  $M$ , there exists a restriction on the uniform norm

$$\|\{f, g\}\| = \sup_{x \in M} |\{f, g\}(x)|,$$

when we perturb  $f, g$  in the uniform norm. The first result in this direction was obtained by F. Cardin and C. Viterbo [CV], who showed that if  $\{f, g\}$  is not identically zero, then

$$\liminf_{\|F-f\|, \|G-g\| \rightarrow 0} \|\{F, G\}\| > 0.$$

This result was improved by M. Entov, L. Polterovich, F. Zapolsky ([EPZ], [Z], [EP1]). It was shown in [EP1], that in fact, for any symplectic manifold  $(M, \omega)$  and any compactly supported  $f, g$ , we have

$$\liminf_{\|F-f\|, \|G-g\| \rightarrow 0} \|\{F, G\}\| = \|\{f, g\}\|.$$

In both statements the functions  $F, G$  are compactly supported.

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We introduce a new approach to the  $C^0$ -rigidity phenomenon. Our main result is summarized in Theorem 1.1.4. Under the assumption that  $\max\{f, g\}$  exists, we provide an explicit lower estimate for the  $\sup\{F, G\}$ , when the functions  $F, G : M \rightarrow \mathbb{R}$  are  $C^0$ -close to  $f, g$  respectively.

The statement of Theorem 1.1.2 coincides with the abovementioned result from [EP1], while stated under slightly more general conditions. In this case, our approach enables us to provide a short proof of the statement.

In order to state the next theorem, we introduce the following definition.

**DEFINITION 1.1.1.** *Let  $(M, \omega)$  be a symplectic manifold. We denote by  $\mathcal{H}^b(M, \omega)$  the set of all smooth functions  $H : M \rightarrow \mathbb{R}$ , such that the Hamiltonian flow generated by  $H$  is complete, that is, the solution exists for any finite time.*

**Theorem 1.1.2.** *Let  $(M, \omega)$  be a symplectic manifold. Then, for any  $f, g \in C^\infty(M)$ ,*

$$\liminf_{F, G \in C^\infty(M), G \in \mathcal{H}^b(M, \omega), \|F - f\|, \|G - g\| \rightarrow 0} \sup\{F, G\} = \sup\{f, g\}.$$

The method of the proof of Theorem 1.1.2 is based on the positivity of the displacement energy of an open subset in  $M$  (see [MS]).

**DEFINITION 1.1.3.** *Let  $(M, \omega)$  be a symplectic manifold. Given a pair of smooth functions  $f, g \in C^\infty(M)$ , we define*

$$\Upsilon_{f,g}^+(\varepsilon) := \sup\{f, g\} - \inf_{F, G \in C^\infty(M), G \in \mathcal{H}^b(M, \omega), \|F - f\| \leq \varepsilon, \|G - g\| \leq \varepsilon} \sup\{F, G\},$$

$$\Upsilon_{f,g}(\varepsilon) := \|\{f, g\}\| - \inf_{F, G \in C^\infty(M), G \in \mathcal{H}^b(M, \omega), \|F - f\| \leq \varepsilon, \|G - g\| \leq \varepsilon} \|\{F, G\}\|.$$

Then we have

**Theorem 1.1.4.** *Let  $(M, \omega)$  be a symplectic manifold. Assume that  $f, g \in C^\infty(M)$  are such that  $\{f, g\}$  attains its maximum at some  $x \in M$ . Assume, in addition, that  $x$  is not a critical point for the functions  $f, g$ . Then*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 6 \left( -\{\{\{f, g\}, f\}, f\}(x) - \{\{\{f, g\}, g\}, g\}(x) \right)^{1/3}.$$

Let us mention that, in the case of a closed manifold  $(M, \omega)$ , the condition that  $x$  is not a critical point for the functions  $f, g$  is satisfied automatically, if we assume that  $\{f, g\}$  is not identically zero.

As will be seen from the proof of Theorem 1.1.4, the expression

$$-\{\{\{f, g\}, f\}, f\}(x) - \{\{\{f, g\}, g\}, g\}(x)$$

is non-negative, provided that the function  $\{f, g\}$  attains its maximum at the point  $x$ .

In the proof of Theorem 1.1.4 we use lower estimates for the symplectic displacement energy. We use the notation  $e(W)$  for the symplectic displacement energy of the set  $W$ .

For our purposes the following weak estimate will suffice.

**PROPOSITION 1.1.5.** *Assume that we have a symplectic embedding*

$$i : U \subset (\mathbb{R}^{2n}, \omega_{std}) \hookrightarrow (M, \omega).$$

Consider a subset  $V \subseteq U$  of the form  $V = Q_1 \times Q_2 \times \cdots \times Q_n$ , where  $Q_1, Q_2, \dots, Q_n \subset \mathbb{R}^2$  are simply connected planar domains. Then we have

$$e(i(V)) \geq \frac{1}{2} \min(\text{Area}(Q_1), \text{Area}(Q_2), \dots, \text{Area}(Q_n)).$$

The Proposition 1.1.5 follows from the inequality (see [MS])

$$e(A) \geq \frac{1}{2} w_G(A)$$

between the displacement energy  $e(A)$  of  $A$ , and the Gromov width

$$w_G(A) = \sup \{ \pi r^2 \mid B^{2n}(r) \text{ embeds symplectically in } A \},$$

where  $B^{2n}(r) \subset \mathbb{R}^{2n}$  is the standard Euclidean ball of radius  $r$ .

It is easy to see that replacing the functions  $f, F$  by  $-f, -F$  in Theorems 1.1.2 and 1.1.4, we will get the analogous statements concerning the  $C^0$ -rigidity of the infimum of the Poisson bracket. Both the rigidity of the supremum and of the infimum imply the corresponding rigidity result for the uniform norm  $\|\{f, g\}\|$  of the Poisson bracket, since we have

$$\|\{f, g\}\| = \max \left( -\inf_M \{f, g\}, \sup_M \{f, g\} \right).$$

The coefficient 4 in the statement of the Theorem 1.1.4 is not the exact value, and can be slightly improved using our method. On the other hand, weaker lower estimates of the form

$$e(i(V)) \geq c \min(\text{Area}(Q_1), \text{Area}(Q_2), \dots, \text{Area}(Q_n))$$

for the displacement energy, will affect only this coefficient, which will become larger. The precise optimal value is still to be found.

It turns out that the estimate on  $\Upsilon_{f,g}^+(\varepsilon)$  in the Theorem 1.1.4 is sharp, up to some constant factor. To obtain a lower bound for  $\Upsilon_{f,g}^+(\varepsilon)$ , we first prove the following local result.

**Theorem 1.1.6.** *Let  $(M, \omega)$  be a symplectic manifold. Assume that we have  $f, g \in C^\infty(M)$ . Denote by  $\Phi : M \rightarrow \mathbb{R}$  the function*

$$\Phi = -\{\{\{f, g\}, f\}, f\} - \{\{\{f, g\}, g\}, g\}.$$

*Assume that  $\{f, g\}$  attains its maximum at the point  $x \in M$ , which is moreover a non-degenerate critical point of  $\{f, g\}$ . Consider a neighborhood  $U$  of  $x$ , and assume that*

$$\{f, g\}(y) < \{f, g\}(x),$$

*for every  $y \in \overline{U} \setminus \{x\}$ . Then we can find a neighborhood  $V$  of  $x$ ,  $\overline{V} \subset U$ , such that for small  $\varepsilon > 0$  there exist smooth functions  $F, G : M \rightarrow \mathbb{R}$ , satisfying*

$$\|F - f\| \leq \varepsilon, \quad \|G - g\| \leq \varepsilon,$$

$$\{F, G\}(y) \leq \{f, g\}(x) - \frac{1}{3} \Phi(x)^{1/3} \varepsilon^{2/3}, \quad \forall y \in U,$$

*and such that  $F = f, G = g$  on  $M \setminus V$ .*

As a result of Theorems 1.1.4, 1.1.6, we obtain the following global result on a closed manifold  $M$ .

**Theorem 1.1.7.** *Let  $(M, \omega)$  be a closed symplectic manifold. Assume that we have  $f, g \in C^\infty(M)$ . Denote by  $\Phi : M \rightarrow \mathbb{R}$  the function*

$$\Phi = -\{\{\{f, g\}, f\}, f\} - \{\{\{f, g\}, g\}, g\}.$$

*Assume that  $x = x_1, x_2, \dots, x_N$  are all the points  $x \in M$  for which  $|\{f, g\}(x)| = \|\{f, g\}\|$ , and assume that all of them are non-degenerate critical points of the function  $\{f, g\}$ . Denote*

$$C = C(f, g) = \min(|\Phi(x_1)|, |\Phi(x_2)|, \dots, |\Phi(x_N)|)^{1/3}.$$

*Then*

$$\frac{1}{3}C \leq \liminf_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}(\varepsilon)}{\varepsilon^{2/3}} \leq \limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}(\varepsilon)}{\varepsilon^{2/3}} \leq 6C.$$

It was shown in [Z], that in the case of dimension 2, if  $\max_M \{f, g\}$  is attained, then the statement of Theorem 1.1.2 in the dimension 2 case becomes local in the sense of section 3 below, and does not require the condition of  $G \in \mathcal{H}^b(M, \omega)$ . However, for dimensions bigger than 2, the situation changes. It turns out that the assumption  $G \in \mathcal{H}^b(M, \omega)$  in Theorems 1.1.2, 1.1.4 is essential. We show this in Example 3.0.10 provided in section 3. Moreover, Example 3.0.11 in section 3 shows the non-locality of Theorem 1.1.4 for any symplectic manifold  $(M, \omega)$ , with  $\dim(M) > 2$ . Examples 3.0.10, 3.0.11 are closely related, and we refer the reader to section 3 for a detailed explanation of these phenomena.

After establishing these results, the statement of Theorem 1.1.4 was re-proved by Entov and Polterovich [EP2], with the use of their own approach.

**1.2 Conditions for the continuity of the Poisson bracket in the uniform norm.** Here we provide another application of the method, used to prove Theorems 1.1.2, 1.1.4. It is natural to ask the following:

**QUESTION 1.2.1.** Suppose we have a symplectic manifold  $(M, \omega)$ , functions  $f, g, h \in C^\infty(M)$ , and sequences

$$f_1, f_2, \dots, g_1, g_2, \dots \in C^\infty(M),$$

such that  $f_n \rightarrow f$ ,  $g_n \rightarrow g$ ,  $\{f_n, g_n\} \rightarrow h$  uniformly on  $M$ . Is it true that  $h = \{f, g\}$ ?

The answer in the general case is negative, as we see from the following example due to Polterovich.

**EXAMPLE 1.2.2.** On the plane  $\mathbb{R}^2$  consider the following sequence of functions:

$$F_n(q, p) = \frac{\chi(p)}{\sqrt{n}} \cos(nq), \quad G_n(q, p) = \frac{\chi(p)}{\sqrt{n}} \sin(nq),$$

where  $\chi \in C^\infty(\mathbb{R})$  given. Then  $\{F_n, G_n\} = \chi(p)\chi'(p)$ , while  $F_n, G_n \rightarrow 0$  uniformly.

We provide a sufficient condition under which we have an affirmative answer to this question.

Let us first introduce the notation needed for the formulation of the theorems in this section.

**DEFINITION 1.2.3.** Suppose we have a smooth manifold  $X$  endowed with a Riemannian metric  $\rho$  and a smooth function  $h : X \rightarrow \mathbb{R}$ . Take an integer  $k \geq 1$ .

For any  $x \in X$ ,  $v \in T_x X$ , with the unit norm  $\|v\|_\rho = 1$ , take a small  $\rho$ -geodesic  $\gamma : [0, \varepsilon) \rightarrow X$ , such that  $\gamma(0) = x, \dot{\gamma}(0) = v$ . Then we denote

$$\|h\|_{x,v,1} := \left| \frac{d}{dt} \Big|_{t=0} h(\gamma(t)) \right|.$$

Next, for  $x \in X$ , denote

$$\|h\|_{x,1} := \max_{v \in T_x X, \|v\|_\rho=1} \|h\|_{x,v,1}.$$

For a given subset  $Y \subset X$  with compact closure  $\bar{Y} \subset X$ , we denote

$$\|h\|_{Y,1} := \sup_{x \in Y} \|h\|_{x,1}.$$

For a given subset  $Y \subset X$  with compact closure  $\bar{Y} \subset X$ , we denote

$$\|h\|_Y := \sup_{x \in Y} |h(x)|.$$

We use the notation  $\text{dist}_\rho(x, y)$  for the  $\rho$ -distance between a pair of points  $x, y \in X$ .

We first prove

**Theorem 1.2.4.** *Let  $(M, \omega)$  be a symplectic manifold, and an open subset  $U \subset M$  with compact closure  $\bar{U} \subset M$ . Assume that we are given a Riemmanian metric  $\rho$  on  $U$ , and smooth functions  $f, g \in C^\infty(M)$ . Then there exists a constant  $C = C(U, \rho, f, g) > 0$ , such that for any  $F_1, G_1, F_2, G_2 \in C^\infty(M)$ , satisfying*

$$\|F_1 - f\|_U, \|F_2 - f\|_U, \|G_1 - g\|_U, \|G_2 - g\|_U < \varepsilon,$$

we have

$$\inf_{y, z \in U} |\{F_1, G_1\}(y) - \{F_2, G_2\}(z)| \leq C\varepsilon \max(1, \|G_1\|_{U,1}, \|G_2\|_{U,1}).$$

As a corollary from Theorem 1.2.4 we obtain

**Theorem 1.2.5.** *Let  $(M, \omega)$  be a symplectic manifold. Assume that we have functions  $f, g, h \in C^\infty(M)$ , and sequences*

$$f_1, f_2, \dots, g_1, g_2, \dots \in C^\infty(M),$$

*such that  $f_n \rightarrow f$ ,  $g_n \rightarrow g$ ,  $\{f_n, g_n\} \rightarrow h$  uniformly on  $M$ . Then if  $\max(\|f_n - f\|_U, \|g_n - g\|_U) \|g_n\|_{U,1} \rightarrow 0$  for any open  $U \subset M$  with compact closure, then  $\{f, g\} = h$ . The norms can be taken with respect to any Riemmanian metric  $\rho$  on  $M$ , and obviously the condition above does not depend on the metric.*

The proof of Theorem 1.2.5 uses Proposition 1.1.5.

As it is easy to see, in Example 1.2.2 we have

$$\max(\|F_n\|, \|G_n\|) \|G_n\|_1 \rightarrow \|\chi\|^2.$$

The result of Theorem 1.2.5 is in the spirit of the work of Humilière [H]. Actually, he provides an affirmative answer to Question 1.2.1, if we assume that the sequences of pairs  $(f_n, g_n)$  of functions belong to some additional structure, namely a pseudo-representation of a normed Lie algebra.

Using Theorem 1.2.4, one can extend the notion of Poisson bracket for some class of non-smooth functions.

DEFINITION 1.2.6. Given a manifold  $X$ , we say that the function  $f : X \rightarrow \mathbb{R}$  is of the Hölder class  $\alpha^+$ , if for some Riemmanian metric  $\rho$  on  $X$  and any  $x \in X$ , we have

$$\lim_{\text{dist}_\rho(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{(\text{dist}_\rho(x,y))^\alpha} = 0.$$

Clearly the definition does not depend on the choice of the metric.

One can show that for given functions  $f, g : M \rightarrow \mathbb{R}$  of the Hölder class  $\frac{1}{2}^+$ , one can define in a canonical way the analog of the Poisson bracket  $\{f, g\}$ , such that for any  $x \in M$ ,  $\{f, g\}(x)$  is not a real number but a closed, finite or infinite interval in  $\mathbb{R}$ .

**1.3 Rigidity for general multi-linear differential operators.** In this subsection we restrict ourselves to compactly supported functions. We ask the following general

QUESTION 1.3.1. For a given smooth manifold  $X^n$ , for which multi-linear differential operators on  $C^\infty(X)$ , either of order 1 or bigger than 1, do we have some sort of  $C^0$ -rigidity?

We concentrate on the following two forms of  $C^0$  rigidity.

DEFINITION 1.3.2. Assume that we have a multi-linear operator

$$B : C^\infty(X)^{\times m} \rightarrow C^\infty(X).$$

On the space  $C^\infty(X)^{\times m}$  consider the following metric: given

$$\mathcal{F} = (f_1, f_2, \dots, f_m), \quad \mathcal{G} = (g_1, g_2, \dots, g_m) \in C^\infty(X)^{\times m},$$

denote

$$d_C(\mathcal{F}, \mathcal{G}) := \max_{1 \leq k \leq m} \|f_k - g_k\|.$$

We say that  $B$  satisfies weak  $C^0$ -rigidity if, for given  $\mathcal{F} \in C^\infty(X)^{\times m}$ , such that  $\|B(\mathcal{F})\| > 0$  we have

$$\liminf_{d_C(\tilde{\mathcal{F}}, \mathcal{F}) \rightarrow 0} \|B(\tilde{\mathcal{F}})\| > 0.$$

We say that  $B$  satisfies strong  $C^0$ -rigidity if, for given  $\mathcal{F} \in C^\infty(X)^{\times m}$ , we have

$$\liminf_{d_C(\tilde{\mathcal{F}}, \mathcal{F}) \rightarrow 0} \|B(\tilde{\mathcal{F}})\| = \|B(\mathcal{F})\|.$$

On one hand, in the case of linear differential operators of the first order, the  $C^0$ -rigidity holds for any such operator, and moreover, it is local. We find an upper bound for the error, and it can be easily shown that it is precise, up to a constant factor. On the other hand, if we consider bilinear differential operators of the first order, then the necessary condition for  $C^0$  rigidity is the anti-symmetry of this form. These statements are summarized in the following:

**Theorem 1.3.3.** Consider a smooth manifold  $X^n$ .

(a) Suppose we are given a differential operator of the first order

$$\lambda : C^\infty(X) \rightarrow C^\infty(X),$$

and a smooth function  $f : X \rightarrow \mathbb{R}$ . Assume that  $\lambda(f)$  attains its maximum at a point  $x$ , such that  $x$  is a non-degenerate critical point of  $\lambda(f)$ . Take an

arbitrary open neighborhood  $U \subset X$  of  $x$ . Then, for any smooth function  $F : X \rightarrow \mathbb{R}$  satisfying  $\|F - f\|_U \leq \varepsilon$ , we have

$$\sup_U \lambda(F) \geq \lambda(f)(x) - \left(\frac{9}{2}\right)^{1/3} (-\lambda^3(f)(x))^{1/3} \varepsilon^{2/3} - O(\varepsilon).$$

(b) Consider a bilinear differential operator of the first order

$$B(\cdot, \cdot) : C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X),$$

which is not antisymmetric. Then there exists a function  $h \in C^\infty(X)$ , and sequences  $f_n, g_n \in C^\infty(X)$  with  $\|f_n - h\|, \|g_n - h\| \rightarrow 0$ , such that  $B(h, h) \neq 0$ ,  $B(f_n, g_n) = 0$ , for every  $n$ .

Let us focus on linear differential operators of the first order. First of all, the error is of the order  $\varepsilon^{2/3}$ , as we had in the case of the Poisson bracket. This appears to be surprising because of the following observation. Given a symplectic manifold  $(M, \omega)$ , and a function  $g \in C^\infty(M)$ , one can define the linear operator  $\lambda(f) := \{f, g\}$ . On the other hand, consider any differential operator of the first order on an even-dimensional manifold  $X$ . Then for any point  $x \in X$ , where the operator does not vanish, there exists a neighborhood  $U$  of  $x$  and a symplectic structure  $\omega$  on  $U$ , such that our differential operator has the form  $\lambda(f) := \{f, g\}$  on  $U$ .

As we see, in Theorem 1.1.4(a) we have freedom in perturbing both of the functions  $f, g$ , while the application of Theorem 1.3.3 allows us to perturb only one of the functions; nevertheless, this greater freedom does not decrease the order of the error. Moreover, as an intermediate result in the proof of Theorem 1.1.4, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 144^{1/3} \left( \max_{\theta} P(\theta) \right)^{1/3},$$

where  $P(\theta) = -\{\{f, g\}, \cos(\theta)f + \sin(\theta)g\}, \cos(\theta)f + \sin(\theta)g\}(x)$ . Replace the functions  $f, g$  by

$$\cos(\theta)f + \sin(\theta)g, -\sin(\theta)f + \cos(\theta)g,$$

for the value of  $\theta$ , which gives us the maximum of  $P(\theta)$ . Then the coefficient  $(-\lambda^3(f))^{1/3}$  from Theorem 1.3.3 gives us the exact coefficient for the estimation of the error in Theorem 1.1.4, up to an absolute constant. Also we see from the proof of Theorem 1.1.6, that in the example which we provide there, we perturb only one of the functions.

**QUESTION 1.3.4.** Is it true, that in the case of general multi-linear differential operators of the first order which satisfy the strong version of  $C^0$ -rigidity, we also have this phenomenon? That is, can the example which gives us the best error up to an absolute constant be obtained by perturbing only one of the functions?

As we see, the constant  $2/3$  is not a special symplectic constant. We conjecture, that in fact the order  $\varepsilon^{2/3}$  for the error is correct for any multi-linear differential operator of the first order, which satisfy the strong version of  $C^0$ -rigidity. It is evident from the Theorem 1.3.3, that it will be true, provided the answer to Question 1.3.4 is affirmative.

Now we turn to the case of bi-linear differential operators of the first order. It follows from Theorem 1.3.3 that in order to have some  $C^0$ -rigidity for a bilinear

differential operator of the first order on  $C^\infty(X)$ , it is necessary for this operator to be anti-symmetric. Actually, the statements of Theorems 1.1.2, 1.1.4 show that for a given manifold  $X$ , their  $C^0$ -rigidity results hold for all Poisson brackets derived from some given symplectic structure  $\omega$  on  $X$ , i.e. it holds for all non-degenerate Poisson brackets on  $X$ . However, taking an arbitrary Poisson bracket on  $X$ , not necessarily non-degenerate, i.e. a bilinear operator

$$\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X),$$

which is skew-symmetric, satisfies a Leibnitz rule and the Jacobi identity, the manifold  $X$  is stratified into a disjoint union of symplectic submanifolds, so we can reduce the situation to the non-degenerate case. Therefore, the statements of Theorems 1.1.2, 1.1.4 hold for any Poisson structure on a smooth manifold  $X$ . Observe that taking a Poisson structure  $\{\cdot, \cdot\}$  on a closed manifold  $X$ , and a non-vanishing smooth function  $H(x) \in C^\infty(X)$ , we can define a new bilinear operator  $B(f, g) = H \cdot \{f, g\}$ . Then  $B$  will satisfy a weak form of  $C^0$  rigidity. A priori, we cannot claim that  $B$  should satisfy the strong  $C^0$ -rigidity, because of the non-locality, presented in Example 3.0.11. However, if we assume that  $X$  admits a fibration  $pr : X \rightarrow \mathcal{B}$  such that for any fiber  $Y \subset X$ , the values of  $\{f, g\}|_Y$  depend only on the restrictions  $f|_Y, g|_Y$ , then, taking any positive  $\mathcal{H} : \mathcal{B} \rightarrow \mathbb{R}$ , the form  $B(f, g)(x) = \mathcal{H}(pr(x))\{f, g\}(x)$  will satisfy a strong form of rigidity, as can be easily seen. For example, one can take a 3-dimensional torus  $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$  with coordinates  $(x, y, z) \in \mathbb{T}^3$ , together with a fibration  $\mathbb{T}^3 \rightarrow \mathbb{T}^1$ ,  $(x, y, z) \mapsto z$ , and consider

$$B(f, g) = (\sin(z)^2 + 1)(f_x g_y - f_y g_x).$$

It is easy to see that this particular  $B$  is not the Poisson bracket. As we see, in this construction the form  $B$  is always degenerate.

QUESTION 1.3.5. (a) Is it true that, for closed manifolds the weak  $C^0$  rigidity holds only for multiples of a Poisson bracket by a non-vanishing function?

(b) Is it true that for closed manifolds, in the case of non-degenerate bilinear forms, the strong  $C^0$ -rigidity holds only for Poisson brackets?

Finally, the following example shows the existence of multi-linear operators of order 1, of any number of functions, that satisfy the strong form of the  $C^0$ -rigidity.

EXAMPLE 1.3.6. Given a natural  $m > 1$ , take  $X = \mathbb{R}^m$ , and define  $m$ -linear  $\Phi : C^\infty(X)^m \rightarrow C^\infty(X)$  as follows: taking  $f_1, f_2, \dots, f_m \in C^\infty(X)$ , define  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  by  $F(x) := (f_1(x), f_2(x), \dots, f_m(x))$  and take  $B(f_1, f_2, \dots, f_m)$  to be the Jacobian  $J_F : \mathbb{R}^m \rightarrow \mathbb{R}$ . The strong  $C^0$  rigidity for this  $B$  follows from simple volume considerations.

**1.4 Higher multiplicities of the critical points of  $\{f, g\}$ .** Theorem 1.1.4, applied to the case when the function  $\{f, g\}$  has a degenerate maximum with multiplicity bigger than 2 at the point  $x$ , gives us only

$$\Upsilon_{f,g}^+(\varepsilon) = o(\varepsilon^{2/3}),$$

without saying what is the order of  $\Upsilon_{f,g}^+(\varepsilon)$ . It turns out that, after some modification of the proof of Theorem 1.1.4, we obtain



**Theorem 1.4.1.** *Let  $(M, \omega)$  be a symplectic manifold.*

*Assume that we have  $f, g \in C^\infty(M)$ , such that  $\{f, g\}$  attains its maximum at some  $x \in M$ , and assume that the function  $\{f, g\}$  has multiplicity  $2l$  at the point  $x$ . Assume in addition, that  $x$  is not a critical point for the functions  $f, g$ . Define a differential operator*

$$\mathcal{D}(k) = \{\{k, f\}, f\} + \{\{k, g\}, g\}.$$

*Then*

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2l/(2l+1)}} \leq -9 \left( \frac{1}{2l!} \mathcal{D}^l(\{f, g\})(x) \right)^{1/(2l+1)}.$$

The analogous statement holds also for the case of the infimum.

**REMARK 1.4.2.** Assume that  $M$  is closed. For every  $\varepsilon > 0$ , define a “function”

$$\mathcal{H}_\varepsilon : M \rightarrow \mathbb{R},$$

$$\mathcal{H}_\varepsilon = \{f, g\} + 9 \sum_{l=1}^{\infty} \varepsilon^{\frac{2l}{2l+1}} \left( \frac{1}{2l!} \mathcal{D}^l(\{f, g\}) \right)^{1/(2l+1)}.$$

Since this series of functions does not have to converge, we consider  $\mathcal{H}_\varepsilon$  as a “jet” in the functional space  $C^\infty(M)$ , i.e. an asymptotic series, depending on the parameter  $\varepsilon$ . Then it is easy to see, that Theorem 1.4.1 is equivalent to

$$\inf_{F, G \in C^\infty(M), G \in \mathcal{H}^b(M, \omega), \|F-f\| \leq \varepsilon, \|G-g\| \leq \varepsilon} \|\{F, G\}\| \geq \|\mathcal{H}_\varepsilon\|,$$

as “jets”. By this we mean that for given  $L \geq 1$ , denoting the function

$$\mathcal{H}_{L,\varepsilon} = \{f, g\} + 9 \sum_{l=1}^L \varepsilon^{\frac{2l}{2l+1}} \left( \frac{1}{2l!} \mathcal{D}^l(\{f, g\}) \right)^{1/(2l+1)},$$

which is a truncation of the asymptotic series  $\mathcal{H}_\varepsilon$ , we have

$$\inf_{F, G \in C^\infty(M), G \in \mathcal{H}^b(M, \omega), \|F-f\| \leq \varepsilon, \|G-g\| \leq \varepsilon} \|\{F, G\}\| \geq \|\mathcal{H}_{L,\varepsilon}\| - o(\varepsilon^{2L/(2L+1)}).$$

In this observation, or reformulation of Theorem 1.4.1, we were able to collect all the cases of high multiplicities, and moreover to get rid of considering all the critical points one by one, and instead, to obtain a global inequality, which does not apply to the critical points. However, the asymptotic series  $\mathcal{H}_\varepsilon$  does not seem natural, because of the possible non-smoothness of the functions, which enter in its definition. It would be interesting to find similar, but correct, description of the result of Theorem 1.4.1. Alternatively, it is possible that such a description requires different framework and needs to be written in other terms.

## 2 Proofs of Theorems

*Proof of Theorem 1.1.2.* Let us first describe the main idea of the proof.

We will use the notation  $X_f, X_g, X_F, X_G$  for the Hamiltonian vector fields generated by the Hamiltonians  $f, g, F, G$  and by  $\Phi_f^t, \Phi_g^t, \Phi_F^t, \Phi_G^t$  the corresponding Hamiltonian flows.

We have  $\{f, g\} = df(X_g)$ . Hence, roughly speaking, the value of the Poisson bracket is the rate of change of values of the function  $f$ , computed through the

Hamiltonian flow  $\Phi_g^t$  generated by  $g$ . Assuming that, for some region  $U \subset M$ , we have  $\sup_M\{F, G\} < \inf_U\{f, g\}$ , we will derive that for some small region  $W \subset U$  and for some  $T > 0$ , the values of  $f(\Phi_g^T(W))$  are essentially bigger than those of  $F(\Phi_G^T(W))$ . If  $\|F - f\|$  is small enough, the values of  $f(\Phi_g^T(W))$  will be still much greater than those of  $f(\Phi_G^T(W))$ . Hence, as a conclusion, we will get that the images  $\Phi_g^T(W), \Phi_G^T(W)$  do not intersect, hence the map  $\Phi_g^{-T} \circ \Phi_G^T$  displaces the set  $W$ . Using the positivity of the symplectic energy of  $W$ , and the upper estimate

$$\|\Phi_g^{-T} \circ \Phi_G^T\|_{Hof} \leq 2T\|g - G\|$$

on the Hofer norm, in the case when the norm  $\|g - G\|$  is small enough, we will come to a contradiction with our assumption that  $\sup_M\{F, G\} < \inf_U\{f, g\}$ .

Let us turn now to the precise proof. Denote  $h = \{f, g\}$ . Take any  $x \in M$  and denote  $K = h(x)$ . Assume that, for some  $\delta > 0$ , we have  $\{F, G\} < K - \delta$  on  $M$ , while  $\|f - F\|, \|g - G\| < \varepsilon$ . Here we will fix a specific  $\delta$ , while  $\varepsilon$  will be taken arbitrarily small. For some neighborhood  $U$  of  $x$ , we will have that  $h(y) \geq K - \frac{\delta}{2}$ , for any  $y \in U$ . Pick some  $V \subset U$  and a positive  $T > 0$ , such that for any  $y \in V$ , the flow  $\Phi_g^t(y)$  exists for  $0 \leq t \leq T$  and, moreover,  $\Phi_g^t(y) \in U$  for every  $0 \leq t \leq T$ . Take an arbitrary point  $y \in V$  and define a function  $K(t) = f(\Phi_g^t(y))$ ,  $t \in [0, T]$ . Then we have

$$K'(t) = df(X_g(\Phi_g^t(y))) = \{f, g\}(\Phi_g^t(y)) \geq K - \frac{\delta}{2},$$

for  $t \in [0, T]$ . Therefore,  $f(\Phi_g^T(y)) - f(y) = K(T) - K(0) \geq T(K - \frac{\delta}{2})$ .

On the other hand, given any  $y \in M$ , denote  $L(t) = F(\Phi_G^t(y))$ ,  $t \geq 0$ . Then we have

$$L'(t) = dF(X_G(\Phi_G^t(y))) = \{F, G\}(\Phi_G^t(y)) \leq K - \delta,$$

for  $t \geq 0$ . Hence  $F(\Phi_G^T(y)) - F(y) = L(T) - L(0) \leq T(K - \delta)$ . Since  $\|F - f\| \leq \varepsilon$ , we conclude that  $f(\Phi_G^T(y)) - f(y) \leq T(K - \delta) + 2\varepsilon$ .

Choose a small enough open subset  $W \subset V$  such that we have  $|f(y) - f(z)| \leq \delta T/3$ , when  $y, z \in W$ . Then for any  $y, z \in W$  we have

$$\begin{aligned} f(\Phi_G^T(y)) &\geq T\left(K - \frac{\delta}{2}\right) + f(y) \geq T\left(K - \frac{\delta}{2}\right) - \frac{\delta T}{3} + f(z) \\ &\geq T\left(K - \frac{\delta}{2}\right) - \frac{\delta T}{3} + f(\Phi_G^T(z)) - T(K - \delta) - 2\varepsilon \\ &= f(\Phi_G^T(z)) + \frac{\delta T}{6} - 2\varepsilon. \end{aligned}$$

Assume that  $\varepsilon < \delta T/12$ . Then we will get that  $f(\Phi_G^T(y)) > f(\Phi_G^T(z))$  for any  $y, z \in W$ . Therefore,  $\Phi_G^T(W) \cap \Phi_g^T(W) = \emptyset$ , hence the map  $\Phi_g^{-T} \circ \Phi_G^T$  displaces the set  $W$ . Then, on one hand, the displacement energy  $e(W) > 0$ , on the other hand we have an estimate for the Hofer norm:

$$\|\Phi_g^{-T} \circ \Phi_G^T\|_{Hof} \leq 2T\|g - G\| < 2T\varepsilon.$$

Therefore, we conclude that  $2T\varepsilon > e(W)$ . Observe that the choice of  $W, T$  depends only on  $f, g, x, \delta$ .

As a conclusion, we get that, given  $f, g, \delta$ , and some point  $x \in M$ , there exists an open  $W \subset M$ , and  $T > 0$ , such that for any  $\varepsilon < \min(\delta T/12, e(W)/2T)$  we have that

for any  $F, G$  satisfying  $\|f - F\|, \|g - G\| < \varepsilon$ , we have  $\sup_M \{F, G\} \geq \{f, g\}(x) - \delta$ . Clearly this implies the statement of Theorem 1.1.2.  $\square$

*Proof of Theorem 1.1.4.* The next definition describes the notation that will be used in the proof.

**DEFINITION 2.0.3.** Suppose we have a smooth manifold  $X$  endowed with a Riemannian metric  $\rho$  and a smooth function  $h : X \rightarrow \mathbb{R}$ . Take an integer  $k \geq 1$ . For any  $x \in X$ ,  $v \in T_x X$  with the unit norm  $\|v\|_\rho = 1$ , take a small  $\rho$ -geodesic  $\gamma : [0, \varepsilon) \rightarrow X$ , such that  $\gamma(0) = x, \dot{\gamma}(0) = v$ . Then we denote

$$\|h\|_{x,v,k} := \left| \frac{1}{k!} \frac{d^k}{dt^k} \right|_{t=0} h(\gamma(t)) \Big|.$$

Next, for  $x \in X$  denote

$$\|h\|_{x,k} := \max_{v \in T_x X, \|v\|_\rho = 1} \|h\|_{x,v,k}.$$

For a given subset  $Y \subset X$  with compact closure  $\overline{Y} \subset X$ , we denote

$$\|h\|_{Y,k} := \sup_{x \in Y} \|h\|_{x,k}.$$

For a given subset  $Y \subset X$  with compact closure  $\overline{Y} \subset X$ , we denote

$$\|h\|_Y := \sup_{x \in Y} |h(x)|.$$

Given a vector field  $v$  on  $X$ , we denote by  $\|v\|_x = \|v(x)\|$  the norm of the vector  $v(x) \in T_x X$ , with respect to  $\rho$ . Then for a subset  $Y \subset X$  with compact closure, we denote  $\|v\|_Y = \sup_{x \in Y} \|v\|_x$ .

We use the notation  $\text{dist}_\rho(x, y)$  for the  $\rho$ -distance between a pair of points  $x, y \in X$ .

Note that for any  $Y \subset X$ ,  $\|\cdot\|_{Y,k}$  is not a norm, but rather a pseudo-norm on the space of smooth functions.

We will use the notation  $X_f, X_g, X_F, X_G$  for the Hamiltonian vector fields generated by the Hamiltonians  $f, g, F, G$ , and  $\Phi_f^t, \Phi_g^t, \Phi_F^t, \Phi_G^t$  for the corresponding Hamiltonian flows.

The proof of Theorem 1.1.4 is a generalization of the idea from the proof of Theorem 1.1.2. The proof can be divided into the following parts. First, we consider functions  $f, g, F, G : M \rightarrow \mathbb{R}$ , such that

$$\begin{aligned} \|f - F\|, \|g - G\| &< \varepsilon, \\ \max\{F, G\} &< \max\{f, g\} - \delta. \end{aligned}$$

We take some neighborhood  $U$  of  $x$  in  $M$ , and a Riemannian metric  $\rho$  on  $U$ . We define some region  $W \subset U$ , depending on parameters  $\alpha, r$ , and estimate the value range of the function  $f$  on the images  $\Phi_g^t(W), \Phi_G^t(W)$ . We conclude that, under certain assumptions on  $\varepsilon, \delta, t$  and the parameters  $\alpha, r$ , the images  $\Phi_g^t(W), \Phi_G^t(W)$  do not intersect. Therefore, under these assumptions,  $W$  is displaced by the map  $\Phi_g^{-t} \circ \Phi_G^t$ , hence we obtain

$$2\varepsilon t > 2\|g - G\|t \geq \|\Phi_g^{-t} \circ \Phi_G^t\|_{\text{Hof}} \geq e(W).$$

On the other hand, we find lower estimates for the displacement energy  $e(W)$  in terms of  $\alpha, r$ . Hence, under the assumptions on  $\varepsilon, \delta, t, \alpha, r$  above, and that

$$\begin{aligned} \|f - F\|, \|g - G\| &< \varepsilon, \\ \max\{F, G\} &< \max\{f, g\} - \delta, \end{aligned}$$

we obtain an inequality concerning  $\varepsilon, \delta, t, \alpha, r$ .

In the next step we consider  $f, g, F, G$ , that satisfy

$$\|f - F\|, \|g - G\| < \varepsilon,$$

and we assume that we have such  $\delta, t, \alpha, r$ , so that the abovementioned assumption is satisfied, but the inequality derived from the energy-capacity argument is not. Then we will have to conclude that

$$\max\{F, G\} \geq \max\{f, g\} - \delta.$$

The next step in the proof is to choose optimal  $t, \alpha, r$  to minimize  $\delta$ . The resulting formula involves estimations of  $C^2, C^1$  norms of  $\{f, g\}, f, g$  on  $U$ , with respect to the metric  $\rho$ . Then we shrink the neighborhood  $U$  to the point  $x$ , arriving to the upper estimate for  $\delta$ , involving the norm of the Hessian of  $\{f, g\}$ , and norms of  $X_f, X_g$  at the point  $x$  with respect to the metric  $\rho$ .

Finally, we choose the optimal metric  $\rho$  to obtain the statement of the Theorem 1.1.4.

Let us turn to the proof. First of all, note that  $x$  is not a critical point for the functions  $f, g$ , and therefore

$$df|_x, dg|_x, X_f(x), X_g(x) \neq 0.$$

We start by choosing a Darboux neighborhood  $i : U \hookrightarrow (M, \omega)$  of  $x$ , where  $0 \in U \subset (\mathbb{R}^{2n}, \omega_{std})$ , and  $i(0) = x$ . Fix an arbitrary Riemannian metric  $\rho$  on  $i(U)$ . Replacing  $U$  by some smaller open subset, we can guarantee that every point in  $i(U)$  can be joint to  $x$  by a  $\rho$ -geodesic, which lies in  $i(U)$ .

Then there exists an open neighborhood  $V \subset U$  of 0, and a positive  $T > 0$ , such that for any  $y \in i(V)$ , the flow  $\Phi_g^t(y)$  exists when  $0 \leq t \leq T$ , and moreover,  $\Phi_g^t(y) \in i(U)$  for every  $0 \leq t \leq T$ . Take some  $0 < r < \text{dist}_\rho(x, M \setminus i(V))$  and some real  $\alpha > 0$ , and consider the set

$$W = W_{r, \alpha} = B_x(r) \cap \{y \in M \mid f(x) < f(y) < f(x) + \alpha\} \subset M,$$

where  $B_x(r)$  is a ball of radius  $r$  centered at  $x$ , with respect to the metric  $\rho$ .

For  $y \in W$ , denote  $K(t) = f(\Phi_g^t(y))$ ,  $t \in [0, T]$ . Then

$$K'(t) = df(X_g(\Phi_g^t(y))) = \{f, g\}(\Phi_g^t(y)).$$

Denoting  $h = \{f, g\}$ , we obtain that

$$f(\Phi_g^t(y)) - f(y) = K(t) - K(0) = \int_0^t K'(s) ds = \int_0^t h(\Phi_g^s(y)) ds.$$

Let us estimate the value  $h(\Phi_g^s(y))$  from below. First of all, we have

$$\text{dist}_\rho(x, \Phi_g^s(y)) \leq \text{dist}_\rho(x, y) + \text{dist}_\rho(y, \Phi_g^s(y)).$$

We have  $\text{dist}_\rho(y, \Phi_g^s(y)) \leq s \|X_g\|_U$ ,  $\text{dist}_\rho(x, y) < r$ , for  $y \in W$ . Hence  $\text{dist}_\rho(x, \Phi_g^s(y)) < r + s \|X_g\|_U$ .

LEMMA 2.0.4. For any  $z \in U$  we have

$$h(z) \geq h(x) - \|h\|_{U,2} \operatorname{dist}_\rho(x, z)^2.$$

*Proof of Lemma 2.0.4.* Take a  $\rho$ -geodesic  $\gamma : [0, a] \rightarrow U$ , such that

$$\gamma(0) = x, \quad \gamma(a) = z,$$

where  $a = \operatorname{dist}_\rho(x, z)$ . Define

$$\phi : [0, a] \rightarrow \mathbb{R}$$

as  $\phi(s) := h(\gamma(s))$ . Then, since the point  $x$  is a maximum point of  $h$ , we have  $\phi'(0) = 0$ . Therefore,

$$h(z) - h(x) = \phi(a) - \phi(0) = \int_0^a \phi'(s) ds = \int_0^a (a-s) \phi''(s) ds.$$

On the other hand,  $|\phi''(s)| \leq 2\|h\|_{U,2}$ , so

$$|h(z) - h(x)| \leq 2\|h\|_{U,2} \int_0^a (a-s) ds = \|h\|_{U,2} a^2 = \|h\|_{U,2} \operatorname{dist}_\rho(x, z)^2,$$

what implies the lemma.  $\square$

Hence for  $t \in [0, T]$  we have

$$\begin{aligned} f(\Phi_g^t(y)) - f(y) &= \int_0^t h(\Phi_g^s(y)) ds \\ &> \int_0^t h(x) - \|h\|_{U,2} (r + s\|X_g\|_U)^2 ds = h(x)t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3, \end{aligned}$$

so

$$f(\Phi_G^t(y)) - f(y) > h(x)t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3. \quad (1)$$

Assume that we have smooth  $F, G : M \rightarrow \mathbb{R}$  and positive  $\varepsilon, \delta > 0$ , such that

$$\|f - F\|, \|g - G\| < \varepsilon$$

and

$$\sup_M \{F, G\} < \max\{f, g\} - \delta = h(x) - \delta.$$

Take some  $z \in M$ , and consider the function  $L(t) = F(\Phi_G^t(z))$ ,  $t \geq 0$ . We have

$$L'(t) = dF(X_G(\Phi_G^t(z))) = \{F, G\}(\Phi_G^t(z)) < h(x) - \delta,$$

hence we get an estimate

$$L(t) - L(0) = F(\Phi_G^t(z)) - F(z) < (h(x) - \delta)t,$$

which holds for any  $z \in M$ ,  $t > 0$ . Since we have  $\|F - f\| < \varepsilon$ , we obtain

$$f(\Phi_G^t(z)) - f(z) < (h(x) - \delta)t + 2\varepsilon. \quad (2)$$

In addition, for any  $y, z \in W$  we have

$$|f(y) - f(z)| < \alpha. \quad (3)$$

From the inequalities (1), (2), (3) we derive, that for any  $y, z \in W$  we have

$$\begin{aligned} f(\Phi_g^t(y)) &> f(y) + h(x)t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3 \\ &> f(z) - \alpha + h(x)t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3 \end{aligned}$$

$$\begin{aligned}
&> f(\Phi_G^t(z)) - (h(x) - \delta)t - 2\varepsilon - \alpha + h(x)t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3 \\
&= f(\Phi_G^t(z)) + \delta t - \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3 - 2\varepsilon - \alpha.
\end{aligned}$$

If we assume that

$$\delta t \geq \frac{1}{3} \frac{\|h\|_{U,2}}{\|X_g\|_U} (r + t\|X_g\|_U)^3 + 2\varepsilon + \alpha \quad (4)$$

holds, then for any  $y, z \in W$  we have

$$f(\Phi_g^t(y)) > f(\Phi_G^t(z)),$$

therefore, the sets  $\Phi_g^t(W), \Phi_G^t(W)$  do not intersect. Hence the map  $\Phi_g^{-t} \circ \Phi_G^t$  displaces  $W$ . We have the following estimate for the Hofer norm:

$$\|\Phi_g^{-t} \circ \Phi_G^t\|_{\text{Hofer}} \leq 2t\|g - G\| < 2\varepsilon t.$$

As a conclusion, we have the following:

**LEMMA 2.0.5.** *Assume now that we have smooth  $F, G : M \rightarrow \mathbb{R}$  and positive  $\varepsilon, \delta > 0$  such that*

$$\|f - F\|, \|g - G\| < \varepsilon$$

and

$$\sup_M \{F, G\} < \max\{f, g\} - \delta = h(x) - \delta.$$

In addition, assume that (4) holds for some

$$0 < t \leq T, \quad 0 < r < \text{dist}_\rho(0, \partial V), \quad 0 < \alpha.$$

Then for the set

$$W = W_{r,\alpha} = B_x(r) \cap \{y \in M \mid f(x) < f(y) < f(x) + \alpha\} \subset M,$$

we have  $2\varepsilon t > e(W)$ .

Consider the case when we have smooth  $F, G : M \rightarrow \mathbb{R}$ , positive  $\varepsilon, \delta > 0$ , and  $0 < t \leq T$ ,  $0 < r < \text{dist}_\rho(0, \partial V)$ ,  $0 < \alpha$ , such that  $\|f - F\|, \|g - G\| < \varepsilon$ , the inequalities (4) and  $2\varepsilon t \leq e(W)$  hold. Then Lemma 2.0.5 will imply that

$$\sup_M \{F, G\} \geq \max\{f, g\} - \delta.$$

Assume that we have shown the existence of a positive constant  $C > 0$ , such that if  $r, \alpha > 0$  are small enough, and in addition,  $\alpha/r$  is small enough, then we have  $e(W_{r,\alpha}) \geq Cr\alpha$ . Then we will take  $\alpha = 2t\varepsilon/Cr$ , so that  $2\varepsilon t \leq e(W)$ . Then the inequality (4) is equivalent to

$$\delta \geq \frac{\|h\|_{U,2}}{3} \frac{(r + t\|X_g\|_U)^3}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{\alpha}{t} = \frac{\|h\|_{U,2}}{3} \frac{(r + t\|X_g\|_U)^3}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{2\varepsilon}{Cr}. \quad (5)$$

Our choice of  $t, r$  will be of the form  $t = P\varepsilon^{1/3}/\|X_g\|_U$ ,  $r = P\varepsilon^{1/3}$ , for some  $P > 0$ . Then we have

$$\frac{\|h\|_{U,2}}{3} \frac{(r + t\|X_g\|_U)^3}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{2\varepsilon}{Cr} = \left( \frac{8}{3} \|h\|_{U,2} P^2 + 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P} \right) \varepsilon^{\frac{2}{3}}.$$

Consider first the case, when  $\|h\|_{U,2} > 0$ . In this case, the value of  $P$  that minimizes the expression

$$\frac{8}{3} \|h\|_{U,2} P^2 + 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P},$$

equals

$$P = \left( \frac{3 \|X_g\|_U + \frac{1}{C}}{8 \|h\|_{U,2}} \right)^{1/3}.$$

Then, for this  $P$ ,

$$\frac{8}{3} \|h\|_{U,2} P^2 + 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P} = 72^{1/3} \left( \|h\|_{U,2} \left( \|X_g\|_U + \frac{1}{C} \right)^2 \right)^{1/3}.$$

In the case of  $\|h\|_{U,2} = 0$ , we fix arbitrary  $P > 0$ .

Note, that the choice of  $P$  we have made, does not depend on  $\varepsilon$ . We have

$$\begin{aligned} t &= \frac{P}{\|X_g\|_U} \varepsilon^{\frac{1}{3}}, \\ r &= P \varepsilon^{1/3}, \\ \alpha &= \frac{2t\varepsilon}{Cr} = \frac{2}{C\|X_g\|_U} \varepsilon, \\ \frac{\alpha}{r} &= \frac{2}{PC\|X_g\|_U} \varepsilon^{\frac{2}{3}}. \end{aligned}$$

Keeping the chosen value of  $P$  fixed, and taking  $\varepsilon \rightarrow 0$ , we have

$$t, \alpha, r, \frac{\alpha}{r} \rightarrow 0.$$

In particular,  $t \leq T$ ,  $r < \text{dist}_\rho(0, \partial V)$ , when  $\varepsilon$  is small enough. Moreover, for small enough  $\varepsilon$ , the values of  $\alpha, r, \alpha/r$  are small, therefore we can apply Lemma 2.0.6 to our situation.

LEMMA 2.0.6. *For any  $C < 1/\|X_f\|_x$ , we have*

$$e(W_{r,\alpha}) \geq C r \alpha,$$

when  $\alpha, r, \alpha/r \rightarrow 0$ .

*Proof of Lemma 2.0.6.* We have  $W_{r,\alpha} \subset i(U)$ , the Darboux neighborhood of  $x$ . Take the pullback of  $W_{r,\alpha}$ , the function  $f$  and the metric  $\rho$  to  $U \subset (\mathbb{R}^{2n}, \omega_{std})$ , and denote the pullbacks by the same notation  $W_{r,\alpha}, f, \rho$ . Then in  $U$  we have

$$W_{r,\alpha} = B_{\rho,0}(r) \cap \{y \in \mathbb{R}^{2n} \mid f(0) < f(y) < f(0) + \alpha\} \subset \mathbb{R}^{2n}.$$

Denote  $b(\xi, \eta) := \rho|_0(\xi, \eta)$  the bilinear form on  $\mathbb{R}^{2n}$ , which is the restriction of  $\rho$  to the tangent space  $T_0(\mathbb{R}^{2n})$ . Denote  $l = df|_0$  - the differential of  $f$  at the point 0. Then define

$$\widetilde{W}_{r,\alpha} = \{y \in \mathbb{R}^{2n} \mid b(y, y) < r^2\} \cap \{y \in \mathbb{R}^{2n} \mid 0 < l(y) < \alpha\} \subset \mathbb{R}^{2n}.$$

Then, for small  $r, \alpha$ , we have  $(1 - o(1))\widetilde{W}_{r,\alpha} \subseteq W_{r,\alpha} \subseteq (1 + o(1))\widetilde{W}_{r,\alpha}$ . Hence it is enough to establish

$$\frac{e(i(\widetilde{W}_{r,\alpha}))}{r\alpha} \geq \frac{1}{\|X_f(0)\|_\rho} - o(1),$$

when  $r, \alpha, \alpha/r$  are small enough. Moreover, one can find a linear symplectic change of coordinates in  $\mathbb{R}^{2n}$ , such that we will have  $l = df|_0 = a \cdot dx_1$ , for some  $a \in \mathbb{R}$ , where  $(x_1, y_1, \dots, x_n, y_n)$  are coordinates in  $\mathbb{R}^{2n}$ , so it is enough to consider this case

only. Denote  $b_{11} = b(\partial/\partial y_1, \partial/\partial y_1)$ . It is easy to see that for every  $1 > \tau > 0$ , there exists some  $\kappa > 0$ , such that the set

$$\{y \in \mathbb{R}^{2n} \mid b(y, y) < r^2\}$$

contains

$$[-\kappa r, \kappa r] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \times [-\kappa r, \kappa r]^{2n-2} \subset \mathbb{R}^{2n},$$

for any  $r > 0$ . Hence the set  $\widetilde{W}_{r,\alpha}$  contains

$$\begin{aligned} [-\kappa r, \kappa r] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \times [-\kappa r, \kappa r]^{2n-2} \cap \left[0, \frac{\alpha}{a}\right] \times \mathbb{R}^{2n-2} \\ = \left[0, \frac{\alpha}{a}\right] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \times [-\kappa r, \kappa r]^{2n-2}, \end{aligned}$$

for small  $\alpha/r$ . We have that

$$\text{Area} \left( \left[0, \frac{\alpha}{a}\right] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \right) = \frac{2\tau}{a\sqrt{b_{11}}} \alpha r,$$

which is smaller than

$$\text{Area}([- \kappa r, \kappa r] \times [- \kappa r, \kappa r]) = 4\kappa^2 r^2,$$

when  $\alpha/r$  is small enough. Therefore, by Proposition 1.1.5 we have that the displacement energy

$$\begin{aligned} e \left( i \left( \left[0, \frac{\alpha}{a}\right] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \times [-\kappa r, \kappa r]^{2n-2} \right) \right) \\ \geq \frac{1}{2} \text{Area} \left( \left[0, \frac{\alpha}{a}\right] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \right) = \frac{\tau}{a\sqrt{b_{11}}} \alpha r. \end{aligned}$$

Hence

$$e(i(\widetilde{W}_{r,\alpha})) \geq e \left( \left[0, \frac{\alpha}{a}\right] \times \left[-\tau \frac{r}{\sqrt{b_{11}}}, \tau \frac{r}{\sqrt{b_{11}}}\right] \times [-\kappa r, \kappa r]^{2n-2} \right) \geq \frac{\tau}{a\sqrt{b_{11}}} \alpha r.$$

We have

$$a^2 b_{11} = a^2 b \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1} \right) = b \left( a \frac{\partial}{\partial y_1}, a \frac{\partial}{\partial y_1} \right).$$

Since  $df|_0 = a \cdot dx_1$ , then  $X_f(0) = a \frac{\partial}{\partial y_1}$ , therefore

$$a^2 b_{11} = b \left( a \frac{\partial}{\partial y_1}, a \frac{\partial}{\partial y_1} \right) = b(X_f(0), X_f(0)) = \|X_f(0)\|_\rho^2,$$

i.e. the square of the norm of the vector  $X_f(0)$  with respect to the metric  $\rho$ . Therefore,

$$e(i(\widetilde{W}_{r,\alpha})) \geq \frac{\tau}{a\sqrt{b_{11}}} \alpha r = \frac{\tau}{\|X_f(0)\|_\rho} \alpha r,$$

and this holds for any fixed  $0 < \tau < 1$ , when we take  $\alpha, r$  to be small enough. This implies the lemma.  $\square$

Because of Lemma 2.0.6, we can take arbitrary  $C < 1/\|X_f\|_x$ . Then in the case of  $\|h\|_{U,2} > 0$ , we can take

$$\delta = 72^{1/3} \left( \|h\|_{U,2} \left( \|X_g\|_U + \frac{1}{C} \right)^2 \right)^{1/3} \varepsilon^{2/3}.$$



In the case of  $\|h\|_{U,2} = 0$ , for any fixed  $P > 0$ , we can take

$$\delta = 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P} \varepsilon^{2/3}.$$

Summarizing the above considerations, we see that if  $\|h\|_{U,2} > 0$ , then it follows that for any Darboux neighborhood  $i : U \hookrightarrow (M, \omega)$  of  $x$ , and a Riemannian metric  $\rho$  on  $i(U)$  we have

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 72^{1/3} \left( \|h\|_{U,2} \left( \|X_g\|_U + \frac{1}{C} \right)^2 \right)^{1/3}.$$

Since this holds for any  $C < 1/\|X_f\|_x$ , we obtain

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 72^{1/3} (\|h\|_{U,2} (\|X_g\|_U + \|X_f\|_x)^2)^{1/3}.$$

This inequality is correct also in the case of  $\|h\|_{U,2} = 0$ , since then, fixing some specific  $C < 1/\|X_f\|_x$ , we have

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P},$$

for any given  $P > 0$ , and hence

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} = 0$$

in this case.

Fixing the same metric  $\rho$  on  $U$ , but shrinking  $U$  to the point  $x$ , we obtain

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 72^{1/3} (\|h\|_{x,2} (\|X_g\|_x + \|X_f\|_x)^2)^{1/3}. \quad (6)$$

The last step in the proof of the Theorem 1.1.4 is to choose the optimal metric  $\rho$  in the neighborhood of  $x$  in order to minimize the expression on the right-hand side of the inequality (6). From the inequality (6) we see that it is only essential to choose the metric on the tangent space  $T_x M$ .

First consider the case when  $X_f(x), X_g(x) \in T_x M$  are linearly independent. In this case, the metric we choose will satisfy

$$\|\cos(\theta)X_f + \sin(\theta)X_g\|_{\rho,x} = 1, \quad (7)$$

for all  $\theta$ . It is easy to see that for any  $\varsigma > 0$  we can find a metric  $\rho$  satisfying (7), so that we will have

$$\|h\|_{x,2} \leq \max_{\theta} \|h\|_{x, \cos(\theta)X_f + \sin(\theta)X_g, 2} + \varsigma. \quad (8)$$

To do this, take any metric  $\rho$  which satisfies (7), consider some linear complement of the linear subspace  $Sp(X_f, X_g) \subset T_x M$ , and then re-scale  $\rho$  by a sufficiently big factor in the direction of this complement.

Assume now that we have a metric  $\rho$  that satisfies (7), (8). Suppose that for the vector  $v_0 = \cos(\theta_0)X_f + \sin(\theta_0)X_g$  we have

$$\max_{\theta} \|h\|_{x, \cos(\theta)X_f + \sin(\theta)X_g, 2} = \|h\|_{x, v_0, 2}.$$

Then we have

$$\|h\|_{x,2}(\|X_g\|_x + \|X_f\|_x)^2 \leq 4\|h\|_{x,v_0,2} + 4\varsigma.$$

We claim that

$$\|h\|_{x,v_0,2} = -\frac{1}{2}\{\{h, \cos(\theta_0)f + \sin(\theta_0)g\}, \cos(\theta_0)f + \sin(\theta_0)g\}(x).$$

In order to compute  $\|h\|_{x,v_0,2}$ , we have to choose a  $\rho$ -geodesic  $\gamma : [0, \varepsilon) \rightarrow M$ , such that  $\gamma(0) = x$ ,  $\dot{\gamma}(0) = v_0$ , and then

$$\|h\|_{x,v_0,2} = \left| \frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} h(\gamma(t)) \right|.$$

However, since  $h$  has at least order 2 at the point  $x$ , we can only require from  $\gamma$  that  $\dot{\gamma}(0) = v_0$ , without the assumption of being geodesic. In what follows, we can take  $\gamma(t) = \Phi_k^t(x)$ , where  $\Phi_k^t$  is the flow of the Hamiltonian  $k := \cos(\theta_0)f + \sin(\theta_0)g$ . Then, denoting by  $X_k$  the Hamiltonian vector field of the Hamiltonian  $k$ , we have

$$\frac{d}{dt} h(\Phi_k^t(x)) = dh(X_k(\Phi_k^t(x))) = \{h, k\}(\Phi_k^t(x)),$$

hence

$$\begin{aligned} \frac{d^2}{dt^2} h(\Phi_k^t(x)) &= \frac{d}{dt} \{h, k\}(\Phi_k^t(x)) = d\{h, k\}(X_k(\Phi_k^t(x))) \\ &= \{\{h, k\}, k\}(\Phi_k^t(x)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|h\|_{x,v_0,2} &= \left| \left( \frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} h(\gamma(t)) \right) \right| = \left| \frac{1}{2} \{\{h, k\}, k\}(x) \right| \\ &= -\frac{1}{2} \{\{h, \cos(\theta_0)f + \sin(\theta_0)g\}, \cos(\theta_0)f + \sin(\theta_0)g\}(x), \end{aligned}$$

since  $x$  is the point of local maximum of  $h$ . Hence we conclude that, denoting  $P(\theta) = -\{\{h, \cos(\theta)f + \sin(\theta)g\}, \cos(\theta)f + \sin(\theta)g\}(x)$ , we have

$$\|h\|_{x,2}(\|X_g\|_x + \|X_f\|_x)^2 \leq 2 \max_{\theta} P(\theta) + 4\varsigma.$$

So we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} &\leq 72^{1/3} \left( 2 \max_{\theta} P(\theta) + 4\varsigma \right)^{1/3} \\ &= 144^{1/3} \left( \max_{\theta} P(\theta) + 2\varsigma \right)^{1/3}. \end{aligned}$$

Since this holds for any  $\varsigma > 0$ , we obtain

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 144^{1/3} \left( \max_{\theta} P(\theta) \right)^{1/3}.$$

It is easy to see that  $P(\theta) + P(\theta + \frac{\pi}{2}) = -\{\{h, f\}, f\}(x) - \{\{h, g\}, g\}(x)$  for every  $\theta$ , and since  $x$  is a local maximum point of  $h$ , we have  $P(\theta) \geq 0$  for every  $\theta$ . This implies  $\max_{\theta} P(\theta) \leq -\{\{h, f\}, f\}(x) - \{\{h, g\}, g\}(x)$ . Therefore,

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} \leq 144^{1/3} \left( -\{\{h, f\}, f\}(x) - \{\{h, g\}, g\}(x) \right)^{1/3}.$$

It remains to check the case when  $X_f(x), X_g(x) \in T_x M$  are linearly dependent. Suppose for instance that  $X_g = qX_f$ , when  $|q| \leq 1$  (the other case is similar). Take

any metric  $\rho$ , such that  $\|X_f\|_{\rho,x} = 1$ , then take some  $\varsigma > 0$ , and re-scale  $\rho$  along some linear complement of  $\text{Span}(X_f)$ , so that we will have

$$\|h\|_{x,2} \leq \|h\|_{x,X_f,2} + \varsigma. \quad (9)$$

We have

$$\|h\|_{x,X_f,2} = -\frac{1}{2}\{\{h,f\},f\}(x),$$

therefore

$$\|h\|_{x,2}(\|X_g\|_x + \|X_f\|_x)^2 \leq -2\{\{h,f\},f\}(x) + 4\varsigma.$$

Hence

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} &\leq 72^{1/3}(-2\{\{h,f\},f\}(x) + 4\varsigma)^{1/3} \\ &= 144^{1/3}(-\{\{h,f\},f\}(x) + 2\varsigma)^{1/3}. \end{aligned}$$

Since this holds for any  $\varsigma > 0$ , we obtain

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{2/3}} &\leq 144^{1/3}(-\{\{h,f\},f\}(x))^{1/3} \\ &\leq 144^{1/3}(-\{\{h,f\},f\}(x) - \{\{h,g\},g\}(x))^{1/3}. \end{aligned}$$

Since  $144^{1/3} < 6$ , we obtain the desired result.  $\square$

*Proof of Theorem 1.1.6.* Denote by  $X_f, X_g$  the Hamiltonian vector fields generated by Hamiltonians  $f, g : M \rightarrow \mathbb{R}$ . Denote  $h = \{f, g\}$ . Since  $x$  is the local maximum point of  $h$ , we have

$$\{\{h,f\},f\}(x), \{\{h,g\},g\}(x) \leq 0.$$

If  $\{\{h,f\},f\}(x) = \{\{h,g\},g\}(x) = 0$ , there is nothing to prove. Consider the complementary case. Without loss of generality, we can assume that  $\{\{h,g\},g\}(x) < 0$ ,  $\{\{h,g\},g\}(x) \leq \{\{h,f\},f\}(x)$  (in the opposite case, we can apply the Theorem 1.1.6 to the functions  $-g, f$ ). Because of  $\{\{h,g\},g\}(x) < 0$ , we have  $X_g(x) \neq 0$ . Hence, for some small neighborhood  $W \subset U$  of  $x$ , there exists a coordinate  $x_1 : W \rightarrow \mathbb{R}$ , such that  $x_1(x) = 0$ ,  $X_g = \partial/\partial x_1$  on  $W$ . Denote  $H = h_{x_1}$ . Then  $H_{x_1} = \{\{h,g\},g\} \neq 0$ , therefore one can extend  $x_1$  to a coordinate system  $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$  on  $W$ , such that

$$H_{y_1}(x) = H_{x_2}(x) = H_{y_2}(x) = \dots = H_{x_n}(x) = H_{y_n}(x) = 0.$$

Note that this is not necessarily a Darboux coordinate system. Denote  $A = -\{\{h,g\},g\}(x) = -h_{x_1 x_1}(x) > 0$ . Take some  $b > 0$ , such that the cube

$$K = \{(x_1, y_1, x_2, y_2, \dots, x_n, y_n) \mid -b \leq x_1, y_1, x_2, y_2, \dots, x_n, y_n \leq b\}$$

is inside  $W$ . Denote also

$$K' = \{y = (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in K \mid -b/3 \leq x_1 \leq b/3\}.$$

For small  $\varepsilon > 0$ , take a smooth  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $\varphi(t) = \frac{1}{2}A^{1/3}\varepsilon^{2/3}t$  for  $t \in [-A^{-1/3}\varepsilon^{1/3}, A^{-1/3}\varepsilon^{1/3}]$ , that  $\varphi'(t) \geq 0$  when  $t \in [-b/3, b/3]$ , that

$$\varphi'(t) \geq \frac{\max_{y \in K \setminus K'} h(y) - h(x)}{2}$$

for  $t \in [-2b/3, 2b/3]$ ,  $\varphi(t) = 0$  for  $t \in [-b, -2b/3] \cup [2b/3, b]$ , and  $|\varphi(t)| \leq \varepsilon$  for any  $t \in \mathbb{R}$ . Then take some bump function  $\psi : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}$ , such that  $\psi = 1$  on  $\frac{1}{3}K$  and  $\psi = 0$  outside  $\frac{2}{3}K$ , and  $0 \leq \psi \leq 1$  on  $\mathbb{R}^{2n}$ . Then define  $F, G : M \rightarrow \mathbb{R}$  by  $F = f$  on  $M \setminus W$ , and  $F = f - \varphi(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n)$  on  $W$ , and then take  $G = g$  on  $M$ . Note that  $F = f$  on  $W \setminus K$ .

First of all, for any  $y = (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in W$ , we have

$$|f(y) - F(y)| = |\varphi(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n)| \leq |\varphi(x_1)| \leq \varepsilon.$$

For  $y \notin W$  we have  $f(y) - F(y) = 0$ . Therefore,  $\|f - F\| \leq \varepsilon$ . As  $G = g$ , we have  $\|g - G\| = 0 \leq \varepsilon$ . On the other hand, for any function  $k : W \rightarrow \mathbb{R}$ , we have

$$\{k, g\} = dk(X_g) = dk\left(\frac{\partial}{\partial x_1}\right) = k_{x_1}.$$

Therefore, for  $y = (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in W$

$$\{F, G\} = \{f - \varphi\psi, g\} = \{f, g\} - \{\varphi\psi, g\} = h - \varphi'(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n).$$

We wish to show that  $\{F, G\} \leq \{f, g\} - \frac{1}{2}A^{1/3}\varepsilon^{2/3}$  on  $W$ . This is equivalent to

$$\varphi'(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n) \geq h(y) - h(x) + \frac{1}{2}A^{1/3}\varepsilon^{2/3}.$$

Because of the condition

$$h_{x_1x_1}(x) = -A,$$

$$h_{x_1y_1}(x) = h_{x_1x_2}(x) = h_{x_1y_2}(x) = \dots = h_{x_1x_n}(x) = h_{x_1y_n}(x) = 0,$$

and since  $x$  is a non-degenerate critical point of  $h$ , we have that the domain

$$\left\{y \in W \mid h(x) - h(y) \leq \frac{1}{2}A^{1/3}\varepsilon^{2/3}\right\}$$

lies inside the set

$$K'' = \{y = (x_1, y_1, \dots, x_n, y_n) \in W, |x_1| \leq A^{-1/3}\varepsilon^{1/3}\} \cap \frac{1}{3}K,$$

when  $\varepsilon$  is small. For  $y \in K''$ ,

$$\varphi'(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n) = \frac{1}{2}A^{1/3}\varepsilon^{2/3} \geq h(y) - h(x) + \frac{1}{2}A^{1/3}\varepsilon^{2/3}.$$

For  $y \in K' \setminus K''$ ,

$$\varphi'(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n) \geq 0 \geq h(y) - h(x) + \frac{1}{2}A^{1/3}\varepsilon^{2/3}.$$

For  $y \in K \setminus K'$ ,

$$\begin{aligned} \varphi'(x_1)\psi(y_1, x_2, y_2, \dots, x_n, y_n) &\geq \frac{\max_{z \in K \setminus K'} h(z) - h(x)}{2} \\ &\geq h(y) - h(x) + \frac{1}{2}A^{1/3}\varepsilon^{2/3}, \end{aligned}$$

when  $\varepsilon$  is small. Since  $F = f, G = g$  on  $U \setminus K$ , and

$$\sup_{y \in U \setminus K} h(y) < h(x),$$

we have

$$\{F, G\}(y) = \{f, g\}(y) = h(y) \geq h(x) - \frac{1}{2}A^{1/3}\varepsilon^{2/3}$$

for  $y \in U \setminus K$ , when  $\varepsilon$  is small. Hence we have shown that for  $V := \text{int}(K) \subset U$ , for  $\varepsilon$  small enough, there exist smooth  $F, G : M \rightarrow \mathbb{R}$ , such that  $F = f, g = G$  on  $M \setminus V$ , and

$$\|F - f\| \leq \varepsilon, \quad \|G - g\| \leq \varepsilon,$$

$$\{F, G\}(y) \leq \{f, g\}(x) - \frac{1}{2} \left( -\{\{h, g\}, g\}(x) \right)^{1/3} \varepsilon^{2/3}, \quad \forall y \in U.$$

We have

$$\frac{1}{2} \left( -\{\{h, g\}, g\}(x) \right)^{1/3} \varepsilon^{2/3} \geq \frac{1}{2} \left( \frac{1}{2} \Phi(x) \right)^{1/3} \varepsilon^{2/3} > \frac{1}{3} \Phi(x)^{1/3} \varepsilon^{2/3},$$

so we obtain the statement of the theorem.  $\square$

*Proof of Theorem 1.1.7.* Note first that Theorems 1.1.4, 1.1.6 have analogous statements for the infimum, instead of the supremum, which clearly can be derived from these theorems.

We have  $\|\{f, g\}\| > 0$ , since otherwise every point in  $M$  is a degenerate critical point of  $\{f, g\}$ . Then for any  $1 \leq k \leq N$  we have  $\{f, g\}(x_k) \neq 0$ , therefore in particular  $x_k$  is not a critical point for each of the functions  $f, g$ . Therefore, we can apply Theorem 1.1.4, together with the remark at the beginning of the proof, to obtain the inequality

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}(\varepsilon)}{\varepsilon^{2/3}} \leq 6 |\Phi(x_k)|^{1/3}.$$

This is true for any  $1 \leq k \leq N$ , so we obtain the desired upper bound.

Let us prove the lower bound. For any  $1 \leq k \leq N$ , take a neighborhood  $x_k \in U_k \subset M$ , such that  $|\{f, g\}(y)| < |\{f, g\}(x_k)|$ , for every  $y \in \overline{U_k} \setminus \{x_k\}$ . Then Theorem 1.1.6 guarantees that there exist neighborhoods  $x_k \in V_k \subset U_k$ , such that for any  $\varepsilon$  small enough there exist functions  $F_k, G_k : M \rightarrow \mathbb{R}$  satisfying

$$\|F_k - f\| \leq \varepsilon, \quad \|G_k - g\| \leq \varepsilon,$$

$$\{F_k, G_k\}(y) \leq \|\{f, g\}\| - \frac{1}{3} |\Phi(x_k)|^{1/3} \varepsilon^{2/3}, \quad \forall y \in U_k,$$

and such that  $F_k = f, G_k = g$  on  $M \setminus V_k$ .

Define  $F, G : M \rightarrow \mathbb{R}$  as  $F = F_k, G = G_k$  on each of  $U_k$  and  $F = f, G = g$  on  $M \setminus \bigcup_{k=1}^N U_k$ . Then on the union  $\bigcup_{k=1}^N U_k$  we clearly will have  $|\{F, G\}| \leq \|\{f, g\}\| - \frac{1}{3} C \varepsilon^{2/3}$ , and for the set  $M \setminus \bigcup_{k=1}^N U_k$  we have  $\max_{M \setminus \bigcup_{k=1}^N U_k} |\{F, G\}| = \max_{M \setminus \bigcup_{k=1}^N U_k} |\{f, g\}| < \|\{f, g\}\|$ , and does not depend on  $\varepsilon$ . Therefore, for small  $\varepsilon$  we have

$$\|\{F, G\}\| \leq \|\{f, g\}\| - \frac{1}{3} C \varepsilon^{2/3}.$$

This example of  $F, G$  shows that

$$\frac{1}{3} C \varepsilon^{2/3} \leq \Upsilon_{f,g}(\varepsilon). \quad \square$$

*Proof of Theorem 1.2.4.* First of all, consider the case when

$$\|G_1\|_{U,1} = \|G_2\|_{U,1} = 0.$$

In this case, we clearly have  $dG_1 = dG_2 = 0$  on  $U$ , hence

$$\{F_1, G_1\} = \{F_2, G_2\} = 0$$

on  $U$ , and then the desired inequality

$$\inf_{y,z \in U} |\{F_1, G_1\}(y) - \{F_2, G_2\}(z)| \leq C \varepsilon \max(1, \|G_1\|_{U,1}, \|G_2\|_{U,1})$$

is satisfied for any choice of  $C > 0$ .

We are left with the case of

$$\max(\|G_1\|_{U,1}, \|G_2\|_{U,1}) > 0.$$

Denote by  $\Phi_{G_1}^t, \Phi_{G_2}^t$  the Hamiltonian flows corresponding to the Hamiltonians  $G_1, G_2$ . Take some open subset  $V \subset U$ , such that the closure  $\overline{V} \subset U$ . Clearly there exists a constant  $c_1 = c_1(U, V)$ , such that for any

$$0 < t \leq T := \frac{c_1}{\max(\|G_1\|_{U,1}, \|G_2\|_{U,1})},$$

and for any  $y \in V$ , we have that  $\Phi_{G_1}^t(y), \Phi_{G_2}^t(y) \in U$ . Take some  $\delta > 0$ , and assume that we have

$$\inf_{y,z \in U} |\{F_1, G_1\}(y) - \{F_2, G_2\}(z)| > \delta.$$

Then one of the following holds:

either

$$(a) \inf_{z \in U} \{F_2, G_2\}(z) - \sup_{y \in U} \{F_1, G_1\}(y) > \delta,$$

or

$$(b) \inf_{y \in U} \{F_1, G_1\}(y) - \sup_{z \in U} \{F_2, G_2\}(z) > \delta.$$

Assume for instance, that (a) holds. Denote  $K = \sup_{y \in U} \{F_1, G_1\}$ . Fix any  $y \in V$ , and denote  $K(t) := F_1(\Phi_{G_1}^t(y))$ , for  $t \in [0, T]$ . Then for every  $t \in [0, T]$  we have  $K'(t) = \{F_1, G_1\}(\Phi_{G_1}^t(y)) \leq K$ , hence for every  $t \in [0, T]$  we have  $K(t) - K(0) = F_1(\Phi_{G_1}^t(y)) - F_1(y) = \int_0^t K'(s)ds \leq Kt$ . Analogously, for any  $z \in V$ , for any  $t \in [0, T]$  we have  $F_2(\Phi_{G_2}^t(z)) - F_2(z) \geq (K + \delta)t$ . Then we have

$$f(\Phi_{G_1}^t(y)) - f(y) \leq Kt + 2\varepsilon, \quad (10)$$

$$f(\Phi_{G_2}^t(z)) - f(z) \geq (K + \delta)t - 2\varepsilon, \quad (11)$$

for any  $y, z \in V$ . Consider any point  $x \in V$  and for  $r, \alpha > 0$  denote

$$W = W_{r,\alpha} = B_x(r) \cap \{y \in M \mid f(x) < f(y) < f(x) + \alpha\} \subset M,$$

where  $B_x(r)$  is a ball of radius  $r$  centered at  $x$ , with respect to the metric  $\rho$ . Then for small  $r, \alpha$  we have  $W \subset V$ . For any  $y, z \in W_{r,\alpha}$  we have

$$|f(y) - f(z)| < \alpha. \quad (12)$$

From the inequalities (10), (11), (12) we conclude that for any  $y, z \in V$  we have

$$\begin{aligned} f(\Phi_{G_2}^t(z)) &\geq f(z) + (K + \delta)t - 2\varepsilon > f(y) - \alpha + (K + \delta)t - 2\varepsilon \\ &\geq f(\Phi_{G_1}^t(y)) - Kt - 2\varepsilon - \alpha + (K + \delta)t - 2\varepsilon = f(\Phi_{G_1}^t(y)) + \delta t - 4\varepsilon - \alpha. \end{aligned}$$

Therefore, if we assume that

$$\delta t \geq 4\varepsilon + \alpha, \quad (13)$$

we get that  $f(\Phi_{G_2}^t(z)) > f(\Phi_{G_1}^t(y))$  for any  $y, z \in W$ , therefore  $\Phi_{G_1}^t(W) \cap \Phi_{G_2}^t(W) = \emptyset$ , hence the set  $W$  is displaced by the map  $\Phi_{G_1}^{-t} \circ \Phi_{G_2}^t$ . We have the estimation

$$\|\Phi_{G_1}^{-t} \circ \Phi_{G_2}^t\|_{Hof} \leq 2t\|G_2 - G_1\| < 2\varepsilon t$$

of the Hofer norm. On the other hand, as a conclusion from Lemma 2.0.6 (see Definition 2.0.3 for the notation used in the lemma), there exists a constant  $c_2 = c_2(\rho, f, x) > 0$ , such that for small  $r, \alpha, \alpha/r$  we have  $e(W_{r,\alpha}) \geq c_2 r \alpha$ . Therefore, we conclude that for  $t \in [0, T]$ , and small  $r, \alpha, \alpha/r > 0$ , satisfying (13) we have

$$c_2 r \alpha \leq e(W_{r,\alpha}) \leq \|\Phi_{G_1}^{-t} \circ \Phi_{G_2}^t\|_{Hof} < 2\varepsilon t.$$

Hence we conclude that given  $\delta, t, r, \alpha > 0$ , satisfying (a), (13), and  $t \in [0, T]$ , and if  $r, \alpha, \alpha/r$  are small enough, then we have  $c_2 r \alpha < 2\varepsilon t$ . An analogous statement holds also for the condition (b). Therefore, we have

**LEMMA 2.0.7.** *There exist constants  $c_1, c_2 > 0$  such that for any  $\delta > 0$ ,  $0 < t \leq \frac{c_1}{\max(\|G_1\|_{U,1}, \|G_2\|_{U,1})}$ , and small  $r, \alpha, \frac{\alpha}{r} > 0$ , satisfying*

$$\inf_{y,z \in U} |\{F_1, G_1\}(y) - \{F_2, G_2\}(z)| > \delta,$$

$$\delta t \geq 4\varepsilon + \alpha,$$

we have  $c_2 r \alpha < 2\varepsilon t$ .

Fix some small  $r = r_0$ , take

$$t = \frac{c_1}{\max(\|G_1\|_{U,1}, \|G_2\|_{U,1}) + 1},$$

$\alpha = \frac{2\varepsilon t}{c_2 r}$ , and then take  $\delta = \frac{4\varepsilon + \alpha}{t}$ . The value of  $r = r_0$  is already chosen to be small and fixed, and since  $t \leq c_1$ , we have  $\alpha \leq \frac{2c_1}{c_2 r_0} \varepsilon$ ,  $\frac{\alpha}{r} = \frac{\alpha}{r_0} \leq \frac{2c_1}{c_2 r_0^2} \varepsilon$ , that are small if  $\varepsilon$  is small.

Therefore, we can apply Lemma 2.0.7, and obtain

$$\inf_{y,z \in U} |\{F_1, G_1\}(y) - \{F_2, G_2\}(z)| \leq \delta.$$

We have

$$\delta = \frac{4\varepsilon}{t} + \frac{\alpha}{t} = \frac{4}{c_1} \varepsilon \max(\|G_1\|_{U,1}, \|G_2\|_{U,1}) + \left( \frac{4}{c_1} + \frac{2}{c_2 r_0} \right) \varepsilon.$$

Therefore, denoting  $C = \frac{8}{c_1} + \frac{4}{c_2 r_0}$ , we obtain the statement of Theorem 1.2.4.  $\square$

*Proof of Theorem 1.2.5.* Consider any open  $U \subset M$ , with compact closure  $\overline{U} \subset M$ . Take any  $n \in \mathbb{N}$  and apply Theorem 1.2.4 to the functions  $F_1 = f_n$ ,  $G_1 = g_n$ ,  $F_2 = f$ ,  $G_2 = g$ . We will get

$$\inf_{y,z \in U} |\{f_n, g_n\}(y) - \{f, g\}(z)|$$

$$\leq C \cdot \max(\|f_n - f\|_U, \|g_n - g\|_U) \cdot \max(1, \|g\|_{U,1}, \|g_n\|_{U,1}).$$

Hence for some constant  $C'$  we have

$$\inf_{y,z \in U} |\{f_n, g_n\}(y) - \{f, g\}(z)|$$

$$\leq C' \max(\|f_n - f\|_U, \|g_n - g\|_U) \|g_n\|_{U,1} + C' \max(\|f_n - f\|_U, \|g_n - g\|_U).$$

Because of the assumptions of the theorem, the right-hand side converges to 0, when  $n \rightarrow \infty$ . On the other hand, the sequence of functions  $\{f_n, g_n\}$  uniformly converges to the function  $h$ . Therefore, we conclude that

$$\inf_{y,z \in U} |h(y) - \{f, g\}(z)| = 0.$$

This holds for any open  $U \subset M$  with compact closure  $\overline{U} \subset M$ . Then, because the functions  $h, \{f, g\}$  are continuous, we get that  $h(x) = \{f, g\}(x)$  for any point  $x \in M$ .  $\square$

*Proof of Theorem 1.3.3.* (a) Since  $\lambda : C^\infty(X) \rightarrow C^\infty(X)$  is a differential operator of the first order, there exists a vector field  $v \in TX$  such that  $\lambda(f) = df(v)$ . There exists a positive  $T = T(x, U)$ , such that we have a well-defined flow  $\Phi^t(x)$  of  $v$ , for  $t \leq T$ , and moreover  $\Phi^t(x) \in U$ , for  $0 \leq t \leq T$ . Assume that we are given  $\varepsilon > 0$  and a smooth function  $F : M \rightarrow \mathbb{R}$ , such that  $\|f - F\| \leq \varepsilon$ . Denote  $K(t) = f(\Phi^t(x))$ ,  $L(t) = F(\Phi^t(x))$ . Assume for a moment that we have some  $\delta > 0$  such that

$$\lambda(F) \leq \lambda(f)(x) - \delta = K'(0) - \delta$$

on  $U$ . Then  $L'(t) \leq (K'(0) - \delta)$ , hence  $L(t) \leq L(0) + (K'(0) - \delta)t$ , for  $t \leq T$ . Because of the assumption  $\|f - F\| \leq \varepsilon$ , we have

$$\begin{aligned} K(t) &\leq L(t) + \varepsilon \leq L(0) + (K'(0) - \delta)t + \varepsilon \leq \\ &\leq K(0) + (K'(0) - \delta)t + 2\varepsilon, \end{aligned}$$

hence

$$\delta t \leq K(0) + K'(0)t - K(t) + 2\varepsilon.$$

We have

$$K(t) = K(0) + K'(0)t + \frac{1}{2}K''(0)t^2 + \frac{1}{6}K'''(0)t^3 + O(t^4).$$

On the other hand,  $K''(0) = 0$ , since the function  $\lambda(f)$  attains its maximum at the point  $x$ , and we see that

$$\begin{aligned} K'(t) &= df(v(\Phi^t(x))) = \lambda(f)(\Phi^t(x)), \\ K''(t) &= d(\lambda(f))(v(\Phi^t(x))) = \lambda^2(f)(\Phi^t(x)), \\ K'''(t) &= d(\lambda^2(f))(v(\Phi^t(x))) = \lambda^3(f)(\Phi^t(x)). \end{aligned}$$

Therefore,  $\delta t \leq -\frac{1}{6}\lambda^3(f)(x)t^3 + O(t^4) + 2\varepsilon$ , hence

$$\delta \leq -\frac{1}{6}\lambda^3(f)(x)t^2 + \frac{2\varepsilon}{t} + O(t^3),$$

for every  $t \leq T$ . We substitute  $t = t_0 = (-6\varepsilon/\lambda^3(f)(x))^{1/3}$  and we get

$$\delta \leq 3 \left( -\frac{1}{6}\lambda^3(f)(x) \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon) = \left( \frac{9}{2} \right)^{1/3} (-\lambda^3(f))^{1/3} \varepsilon^{2/3} + O(\varepsilon).$$

Note that  $t_0 < T$ , when  $\varepsilon$  is small. This observation leads to the desired result.

(b) Since  $B(\cdot, \cdot)$  is not anti-symmetric, there exists some  $h \in C^\infty(X)$  such that  $B(h, h)$  is a non-zero function. Take any smooth nondecreasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $\varphi(t) = 2n$ ,  $\forall t \in [2n, 2n+1]$ , for every  $n \in \mathbb{Z}$ . Define  $f_n(x) = \frac{1}{n}\varphi(nh(x))$ ,  $g_n(x) = \frac{1}{n}\varphi(nh(x) + 1)$ . It is easy to see, that  $f_n, g_n \rightarrow h$  uniformly, but

$$\begin{aligned} B(f_n, g_n)(x) &= B\left(\frac{1}{n}\varphi(nh(x)), \frac{1}{n}\varphi(nh(x) + 1)\right) \\ &= \varphi'(nh(x))\varphi'(nh(x) + 1)B(h, h) = 0, \end{aligned}$$

since  $\varphi'(t)\varphi'(t+1) = 0 \forall t \in \mathbb{R}$ . □

*Proof of Theorem 1.4.1.* Denote  $h = \{f, g\}$ . The proof goes similarly to that of Theorem 1.1.4. We will use the notation in the Definition 2.0.3, introduced in the proof of Theorem 1.1.4.



Instead of inequality (5) we will have

$$\begin{aligned}\delta &\geq \frac{\|h\|_{U,2l}}{2l+1} \frac{(r+t\|X_g\|_U)^{2l+1}}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{\alpha}{t} \\ &= \frac{\|h\|_{U,2l}}{2l+1} \frac{(r+t\|X_g\|_U)^{2l+1}}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{2\varepsilon}{Cr}.\end{aligned}$$

Our choice of  $t, r$  will be of the form  $t = P\varepsilon^{\frac{1}{2l+1}}/\|X_g\|_U$ ,  $r = P\varepsilon^{\frac{1}{2l+1}}$ , for some  $P > 0$ . Then we have

$$\begin{aligned}\frac{\|h\|_{U,2l}}{2l+1} \frac{(r+t\|X_g\|_U)^{2l+1}}{t\|X_g\|_U} + \frac{2\varepsilon}{t} + \frac{2\varepsilon}{Cr} \\ = \left( \frac{2^{2l+1}}{2l+1} \|h\|_{U,2l} P^{2l} + 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P} \right) \varepsilon^{\frac{2l}{2l+1}}.\end{aligned}$$

We fix  $P$ , that minimizes the expression

$$\frac{2^{2l+1}}{2l+1} \|h\|_{U,2l} P^{2l} + 2 \left( \|X_g\|_U + \frac{1}{C} \right) \frac{1}{P}.$$

The corresponding value of  $P$  does not depend on  $\varepsilon$ . Then we take  $\varepsilon$  small enough, such that the assumptions of Lemma 2.0.6 are satisfied, and we obtain

$$\frac{4l+2}{l} \left( \frac{l}{2l+1} \right)^{\frac{1}{2l+1}} \left( \|h\|_{U,2l} \left( \|X_g\|_U + \frac{1}{C} \right)^{2l} \right)^{\frac{1}{2l+1}} \varepsilon^{\frac{2l}{2l+1}}.$$

Then, by the same arguments as in Theorem 1.1.4 we arrive at

$$\limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{\frac{2l}{2l+1}}} \leq \frac{8l+4}{l} \left( \frac{l}{4l+2} \right)^{\frac{1}{2l+1}} \left( \frac{1}{2l!} \max_{\theta} P_{2l}(\theta) \right)^{\frac{1}{2l+1}},$$

where  $P_{2l}(\theta)$  equals

$$-\{ \dots \{ \{ h, \cos(\theta)f + \sin(\theta)g \}, \cos(\theta)f + \sin(\theta)g \}, \dots, \cos(\theta)f + \sin(\theta)g \}(x),$$

when the Poisson bracket is taken  $2l$  times. Note that  $P_{2l}$  is a non-negative trigonometric polynomial of degree  $\leq 2l$ .

**LEMMA 2.0.8.** *There exists a complex trigonometric polynomial  $Q(\theta)$  of degree  $\leq l$ , such that*

$$P_{2l}(\theta) = |Q(\theta)|^2.$$

*Proof of Lemma 2.0.8.* Let us remark, that along the proof we will only use the fact that  $P_{2l}(\theta)$  is non-negative.

Denoting  $z = \cos(\theta) + i\sin(\theta)$ , the trigonometric polynomial  $P_{2l}(\theta)$  can be written as a polynomial of  $z, 1/z$ , and there exists a complex polynomial  $T \in \mathbb{C}[z]$ , such that

$$P_{2l}(\theta) = \frac{1}{z^r} T(z),$$

and  $T(0) \neq 0$ . Since  $P_{2l}(\theta)$  is a real number for any  $\theta \in \mathbb{R}$ , then for any  $z \in \mathbb{C}$ ,  $|z| = 1$ , we have that  $\frac{1}{z^r} T(z) \in \mathbb{R}$ , hence

$$\frac{1}{z^r} T(z) = \overline{\frac{1}{z^r} T(z)}.$$

Assume that  $T(z) = c \prod_{k=0}^m (z - \alpha_k)$ . Since  $T(0) \neq 0$ , we have that  $\alpha_1, \alpha_2, \dots, \alpha_m \neq 0$ .

Then for any  $z \in \mathbb{C}$  with  $|z| = 1$ , we have

$$\begin{aligned} \overline{\frac{1}{z^r} T(z)} &= \frac{1}{\bar{z}^r} \bar{c} \prod_{k=0}^m (\bar{z} - \bar{\alpha}_k) = z^r \bar{c} \prod_{k=0}^m \left( \frac{1}{z} - \bar{\alpha}_k \right) \\ &= \frac{(-1)^m \bar{c}}{\prod_{k=0}^m \bar{\alpha}_k} z^{r-m} \prod_{k=0}^m \left( z - \frac{1}{\bar{\alpha}_k} \right). \end{aligned}$$

Denote  $c' = \frac{(-1)^m \bar{c}}{\prod_{k=0}^m \bar{\alpha}_k}$ . Then

$$\frac{1}{z^r} T(z) = \overline{\frac{1}{z^r} T(z)} = c' z^{r-m} \prod_{k=0}^m \left( z - \frac{1}{\bar{\alpha}_k} \right),$$

and hence

$$z^m T(z) - c' z^{2r} \prod_{k=0}^m \left( z - \frac{1}{\bar{\alpha}_k} \right) = 0,$$

for any  $z \in \mathbb{C}$  with  $|z| = 1$ . Since a non-zero polynomial must have a finite number of roots, we must have an identity

$$z^m T(z) = c z^m \prod_{k=0}^m (z - \alpha_k) \equiv c' z^{2r} \prod_{k=0}^m \left( z - \frac{1}{\bar{\alpha}_k} \right),$$

as polynomials. Hence the list

$$\frac{1}{\bar{\alpha}_1}, \frac{1}{\bar{\alpha}_2}, \dots, \frac{1}{\bar{\alpha}_m}$$

is a permutation of

$$\alpha_1, \alpha_2, \dots, \alpha_m.$$

Moreover, if some  $\alpha_j$  satisfies  $|\alpha_j| = 1$ , then its multiplicity as a root of the polynomial  $T(z)$ , is even. Indeed, write  $\alpha_j = e^{i\theta_j}$ ,  $\theta_j \in \mathbb{R}$ , and consider the limit

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{P_{2l}(\theta_j + \tau)}{P_{2l}(\theta_j - \tau)} &= \lim_{\tau \rightarrow 0} \frac{\alpha_j^r e^{-ir\tau}}{\alpha_j^r e^{ir\tau}} \lim_{\tau \rightarrow 0} \frac{T(\alpha_j e^{i\tau})}{T(\alpha_j e^{-i\tau})} = \lim_{\tau \rightarrow 0} \frac{T(\alpha_j e^{i\tau})}{T(\alpha_j e^{-i\tau})} \\ &= \lim_{\tau \rightarrow 0} \frac{c \prod_{k=0}^m (\alpha_j e^{i\tau} - \alpha_k)}{c \prod_{k=0}^m (\alpha_j e^{-i\tau} - \alpha_k)} = \prod_{k=0}^m \lim_{\tau \rightarrow 0} \frac{e^{i\tau} - \alpha_k \alpha_j^{-1}}{e^{-i\tau} - \alpha_k \alpha_j^{-1}}. \end{aligned}$$

We have that each of the terms  $\lim_{\tau \rightarrow 0} \frac{e^{i\tau} - \alpha_k \alpha_j^{-1}}{e^{-i\tau} - \alpha_k \alpha_j^{-1}}$  equals 1 if  $\alpha_k \neq \alpha_j$ , and  $-1$  if  $\alpha_k = \alpha_j$ . Therefore, the limit equals 1 if the multiplicity of  $\alpha_k$  is even, and  $-1$  if the multiplicity of  $\alpha_k$  is odd. On the other hand, the limit  $\lim_{\tau \rightarrow 0} \frac{P_{2l}(\theta_j + \tau)}{P_{2l}(\theta_j - \tau)}$  must be non-negative, because the trigonometric polynomial  $P_{2l}$  is non-negative. This proves, that the multiplicity  $\alpha_j$  is even.

As a conclusion, we obtain that the list of roots

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

splits into pairs  $\beta_j, \gamma_j$ ,  $j = 1, 2, \dots, s$ , such that  $\gamma_j = 1/\bar{\beta}_j$ , for every  $1 \leq j \leq s$ , where  $2s = m$ . Denote

$$q(z) := \prod_{k=0}^s (z - \beta_k).$$

Then for  $z = \cos(\theta) + i \sin(\theta)$ , we have

$$\begin{aligned}
q(z)\overline{q(z)} &= \prod_{k=0}^s (z - \beta_k) \overline{\prod_{k=0}^s (z - \beta_k)} \\
&= \prod_{k=0}^s (z - \beta_k) \prod_{k=0}^s \left( \frac{1}{z} - \overline{\beta_k} \right) = \prod_{k=0}^s (z - \beta_k) \prod_{k=0}^s \left( \frac{1}{z} - \overline{\beta_k} \right) \\
&= \left( (-1)^s \prod_{k=0}^s \overline{\beta_k} \right) \frac{1}{z^s} \prod_{k=0}^s (z - \beta_k) \prod_{k=0}^s (z - \gamma_k) \\
&= \left( (-1)^s \prod_{k=0}^s \overline{\beta_k} \right) \frac{1}{z^s} \prod_{k=0}^m (z - \alpha_k) = \frac{(-1)^s \prod_{k=0}^s \overline{\beta_k}}{c} \frac{1}{z^s} T(z) \\
&= \frac{(-1)^s \prod_{k=0}^s \overline{\beta_k}}{c} z^{r-s} P_{2l}(\theta).
\end{aligned}$$

Denote  $c'' := \frac{(-1)^s \prod_{k=0}^s \overline{\beta_k}}{c}$ . Then since we have that  $q(z)\overline{q(z)}, P_{2l}(\theta) > 0$  for any  $\theta$ , except, may be, a finite number of values, therefore  $c'' z^{r-s}$  is a positive real number, for any  $z \in \mathbb{C}$ ,  $|z| = 1$ , possibly except a finite number of values. As a consequence, we have that  $r = s$ , and  $c''$  is a positive real number. Hence

$$q(z)\overline{q(z)} = c'' P_{2l}(\theta),$$

and if we denote  $Q(\theta) := \frac{1}{\sqrt{c''}} q(\cos(\theta) + i \sin(\theta))$ , we obtain

$$|Q(\theta)|^2 = Q(\theta)\overline{Q(\theta)} = P_{2l}(\theta). \quad \square$$

LEMMA 2.0.9.

$$\max_{\theta} P_{2l}(\theta) \leq \frac{2l+1}{2\pi} \int_0^{2\pi} P_{2l}(\theta) d\theta.$$

*Proof of Lemma 2.0.9.* Because of Lemma 2.0.8 there exists a complex trigonometric polynomial  $Q(\theta)$  of degree  $\leq l$ , such that  $P_{2l}(\theta) = |Q(\theta)|^2$ . Denote by  $a_{-l}, a_{-l+1}, \dots, a_l$  the Fourier coefficients of  $Q(\theta)$ . Then by Holder inequality, for any  $\phi$  we have

$$\begin{aligned}
P_{2l}(\phi) &= |Q(\phi)|^2 = |a_{-l}e^{-il\phi} + a_{-l+1}e^{-i(l-1)\phi} + \dots + a_l e^{il\phi}|^2 \\
&\leq (|a_{-l}|^2 + |a_{-l+1}|^2 + \dots + |a_l|^2)(2l+1) = \frac{2l+1}{2\pi} \int_0^{2\pi} |Q(\theta)|^2 d\theta \\
&= \frac{2l+1}{2\pi} \int_0^{2\pi} |Q(\theta)|^2 d\theta = \frac{2l+1}{2\pi} \int_0^{2\pi} P_{2l}(\theta) d\theta. \quad \square
\end{aligned}$$

Assume that  $P_{2l}(\theta) = \sum_{k=0}^{2l} c_k \cos(\theta)^{2l-k} \sin(\theta)^k$ . We have

$$\int_0^{2\pi} \cos(\theta)^{2l-k} \sin(\theta)^k d\theta = 0,$$

when  $k$  is odd, and

$$\int_0^{2\pi} \cos(\theta)^{2l-k} \sin(\theta)^k d\theta = 2\mathbf{B}\left(\frac{k}{2} + \frac{1}{2}, l - \frac{k}{2} + \frac{1}{2}\right),$$

for even  $k$ , where  $\mathbf{B}(x, y)$  is the beta-function. It is easy to see that for any  $0 \leq k \leq 2l$ , we have that  $c_k$  equals the sum of terms of the form

$-\{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{2l}\}(x)$ , when each of the functions  $f_j$  is one of  $f, g$ , while the function  $f$  occurs  $2l - k$  times, and  $g$  occurs  $k$  times. Since  $h$  has multiplicity at least  $2l$  at the point  $x$ , all these terms are equal. Indeed, for any  $1 \leq m < 2l$ , denoting

$$H = \{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{m-1}\},$$

we have

$$\{\{H, f_m\}, f_{m+1}\} = \{\{H, f_{m+1}\}, f_m\} + \{H, \{f_m, f_{m+1}\}\},$$

hence

$$\begin{aligned} & \{\dots\{\{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{m-1}\}, f_m\}, \dots, f_{2l}\} \\ &= \{\dots\{\{\{H, f_m\}, f_{m+1}\}, f_{m+2}\}, \dots, f_{2l}\} \\ &= \{\dots\{\{H, f_{m+1}\}, f_m\}, \dots, f_{2l}\} + \{\dots\{\{H, \{f_m, f_{m+1}\}\}, f_{m+2}\}, \dots, f_{2l}\} \\ &= \{\dots\{\{\{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{m-1}\}, f_{m+1}\}, f_m\}, f_{m+2}\}, \dots, f_{2l}\} \\ &\quad + \{\dots\{\{\{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{m-1}\}, \{f_m, f_{m+1}\}\}, f_{m+2}\}, \dots, f_{2l}\}, \end{aligned}$$

and

$$\{\dots\{\{\{\dots\{\{h, f_1\}, f_2\}, \dots\}, f_{m-1}\}, \{f_m, f_{m+1}\}\}, f_{m+2}\}, \dots, f_{2l}\}(x) = 0,$$

since we have applied the Poisson bracket  $2l - 1$  times, starting with the function  $h$ , and  $h$  has multiplicity  $2l$  at  $x$ . Therefore, we have that

$$c_k = \binom{2l}{k} H_k(x) = \frac{1}{\mathbf{B}(k, 2l - k)} H_k(x),$$

where

$$H_k = -\{\dots\{\{h, f\}, f\}, \dots\}, f\}, g\}, g\}, \dots, g\},$$

when  $f$  appears  $2l - k$  times, and  $g$  appears  $k$  times. From all these observations we have

$$\int_0^{2\pi} P_{2l}(\theta) d\theta = 2 \sum_{m=0}^l \frac{\mathbf{B}(m + \frac{1}{2}, l - m + \frac{1}{2})}{\mathbf{B}(2m, 2l - 2m)} H_{2m}(x).$$

Using the identities, concerning the  $\mathbf{B}$  and  $\Gamma$ -functions, one can check that

$$\frac{\mathbf{B}(m + \frac{1}{2}, l - m + \frac{1}{2})}{\mathbf{B}(2m, 2l - 2m)} = \binom{l}{m}.$$

Again, because  $h$  has multiplicity  $2l$  at  $x$ , we have

$$\sum_{m=0}^l \binom{l}{m} H_{2m}(x) = -\mathcal{D}^l(\{f, g\})(x).$$

Summarizing the above considerations, we get that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{\Upsilon_{f,g}^+(\varepsilon)}{\varepsilon^{\frac{2l}{2l+1}}} &\leq -\frac{8l+4}{l} \left(\frac{l}{4l+2}\right)^{\frac{1}{2l+1}} \left(\frac{2l+1}{\pi}\right)^{\frac{1}{2l+1}} \left(\frac{1}{2l!} \mathcal{D}^l(\{f, g\})(x)\right)^{\frac{1}{2l+1}} \\ &\leq -9 \left(\frac{1}{2l!} \mathcal{D}^l(\{f, g\})(x)\right)^{\frac{1}{2l+1}}. \end{aligned}$$

□

### 3 Non-locality

On first sight it seems that the statement of Theorem 1.1.4 is local, in the sense that if the Poisson bracket  $\{f, g\}$  attains its maximum at the point  $x \in M$ , then for any two sequences

$$f_1, f_2, \dots, g_1, g_2, \dots \in C^\infty(M),$$

such that  $\|f_n - f\| \rightarrow 0$ ,  $\|g_n - g\| \rightarrow 0$ , there exists a sequence  $x_n \rightarrow x$ , such that

$$\liminf_{n \rightarrow \infty} \{f_n, g_n\}(x_n) \geq \max\{f, g\}.$$

In fact, we cannot conclude that, since the flow  $\Phi_G^t$  can be very fast, and during a small time can exit a neighborhood of  $x$ . Actually, the locality does not hold for any dimension  $n > 2$ . For dimension 2 the locality was proved by Zapolsky [Z].

On the other hand, the condition of existence of the flow  $\Phi_G^t$  for all time  $t$  is essential, as we will see in the example below.

The examples that reflect both of the remarks above are based on the example of Polterovich, mentioned in Example 1.2.2.

EXAMPLE 3.0.10. Consider the manifold

$$M = \{(x, y, z, u) \in \mathbb{R}^4 \mid 1 < z < 1\} \subset \mathbb{R}^4,$$

endowed with the standard symplectic form  $\omega = dx \wedge dy + dz \wedge du$ . Let  $\chi(t) := \sqrt{2t+2}$ ,  $t \in (-1, +\infty)$ . Then  $\chi(t)\chi'(t) = 1$ . Consider the functions

$$f(x, y, z, u) = x, \quad g(x, y, z, u) = y,$$

and define

$$f_n(x, y, z, u) = x + \frac{\chi(z)}{\sqrt{n}} \cos(nu), \quad g_n(x, y, z, u) = y - \frac{\chi(z)}{\sqrt{n}} \sin(nu),$$

for  $n = 1, 2, 3, \dots$ . Then  $f_n \rightarrow f$ ,  $g_n \rightarrow g$  uniformly on  $M$ . However, we have  $\{f, g\} \equiv 1$ , but  $\{f_n, g_n\} \equiv 0$  for every  $n$ , so rigidity does not hold in its weakest sense.

The reason is that the flows  $\Phi_{g_n}^t$  are not defined for arbitrary time  $t$ .

As a corollary of Example 3.0.10, we derive the non-locality of Theorem 1.1.4. We already see the non-locality in Example 3.0.10, however,  $g_n$  does not belong to  $\mathcal{H}^b(M, \omega)$ . One can fix this problem by the following truncation of the functions.

EXAMPLE 3.0.11. Consider the manifold  $M$  and functions

$$f, g, f_n, g_n : M \rightarrow \mathbb{R},$$

$n = 1, 2, \dots$ , as in the previous Example 3.0.10. Take a smooth function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $\psi(x) = 1$  for  $|x| \leq 1/4$ ,  $\psi(x) = 0$  for  $|x| \geq 1/3$ , and  $x\psi'(x) \leq 0$  for all  $x$ . Then define  $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}$  by  $\varphi(x, y, z, u) = \psi(x)\psi(y)\psi(z)\psi(u)$ . Then  $x\varphi_x, y\varphi_y \leq 0$ . Denote

$$\begin{aligned} F(p) &= f(p)\varphi(p), & G(p) &= g(p)\varphi(p), \\ F_n(p) &= f_n(p)\varphi(p), & G_n(p) &= g_n(p)\varphi(p), \end{aligned}$$

for  $n = 1, 2, 3, \dots$ , and  $p \in M$ . Then  $F, G, F_n, G_n$  are all compactly supported. We have

$$\{F, G\} = \{f\varphi, g\varphi\} = \varphi^2 + \varphi y\{x, \varphi\} + \varphi x\{\varphi, y\} = \varphi^2 + \varphi y\varphi_y + \varphi x\varphi_x \leq \varphi^2 \leq 1$$

at every point, and  $\{F, G\} = 1$  in the cube  $K := \{|x|, |y|, |z|, |u| < 1/4\}$ . However, for every  $p \in K$ , we have the equality  $F_n = f_n, G_n = g_n$ , hence  $\{F_n, G_n\} = 0$  in  $K$ . This reflects the non-locality. Note that

$$\text{supp } F, G, F_n, G_n \subset \{|x|, |y|, |z|, |u| \leq \tfrac{1}{3}\}.$$

Hence non-locality holds for any symplectic manifold of dimension 4, because of the existence of a Darboux chart on  $M$ , and re-scaling of  $F, G, F_n, G_n$ , in order that their supports be contained in this chart. Surely this is true in any dimension of  $M$ , since one can provide a similar example for any even dimension bigger than 4.

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